Variation of Energy

(1) Energys:

For R-m (M^n, g), $p, q \in M$. We'd like to find shortest curve joining p to q. i.e.

- Find $\gamma \in C_{pq} := \{ \gamma : [a, b] \to M \mid \gamma(a) = p, \gamma(b) = q, \gamma$ piecewise smooth$\}$ to minize:

$L(\gamma) := \int_a^b |\gamma'(t)| dt.$

Def: Dirichlet energy of curve γ is

$$E(\gamma) := \frac{1}{2} \int_a^b \langle \gamma', \gamma' \rangle dt$$

prop. A curve γ minizes $E(\gamma) \iff$ it minizes $L(\gamma)$ and $|\gamma'|$ is const.

Pf: $L(\gamma)^2 \overset{\text{Cauchy}}{\leq} (\int_a^b 1^2)(\int_a^b |\gamma'(t)|^2 dt)$

Rmk: γ can be reparametrial to have const. $|\gamma'|$. So:

$\min L(\gamma) \iff \min E(\gamma)$.

Def: i) A variation of γ is smooth map

$\varphi : [a,b] \times (-\varepsilon, \varepsilon) \to M$. s.t. $\varphi_t(s) :=$ $\varphi(s,t)$ is one-parameter family of curves and $\varphi(s,0) = \varphi_0(s) = \varphi(s)$

ii) Denote $\varphi_s := D\varphi(\partial s)$, $\varphi_t := D\varphi(\partial t)$ velocity of $\varphi_t(\cdot)$ and variation field along $\varphi_0(s)$ resp. We have: $\varphi_s(s,0) = \varphi'(s)$ and denote $V(s) = V_{\varphi_s} := \varphi_t(s,0) \in T_{\varphi(s)}M$.

<u>Lemma</u> $\nabla_{\varphi_s}\varphi_t = \nabla_{\varphi_t}\varphi_s$.

<u>Pf</u>: $0 = D\varphi([\partial s, \partial t]) = [\varphi_s, \varphi_t]$
$$\overset{LC}{=} \nabla_{\varphi_s}\varphi_t - \nabla_{\varphi_t}\varphi_s.$$

<u>Thm.</u> (First variation of Energy)
Given any variation of curve φ with variation field V. The 1^{st} varia. energy is $\delta_V E(\varphi) := \frac{d}{dt}\Big|_{t=0} E(\varphi_t)$
$$= \langle V, \varphi' \rangle \Big|_a^b - \int_a^b \langle V, \nabla_{\varphi'}\varphi' \rangle ds$$

<u>Pf</u>: $\frac{d}{dt} E(\varphi_t) = \frac{1}{2}\int_a^b \frac{\partial}{\partial t} \langle \varphi_s, \varphi_s \rangle ds$
$$= \frac{1}{2}\int_a^b \varphi_t \langle \varphi_s, \varphi_s \rangle ds$$

$\overset{LC}{=} \int_a^b \langle \nabla_{\varphi_t} \varphi_s, \varphi_s \rangle ds$

$\overset{\text{Lem}}{\underset{LC}{=}} \int_a^b \frac{\partial}{\partial s} \langle \varphi_t, \varphi_s \rangle - \langle \varphi_t, \nabla_{\varphi_s} \varphi_s \rangle ds$

$= \langle \varphi_t, \varphi_s \rangle |_a^b - \int_a^b \langle \varphi_t, \nabla_{\varphi_s} \varphi_s \rangle ds$

<u>Cor.</u> (First variation of length.)

$\partial L(\gamma) / \partial t |_{t=0} = - \int_a^b \langle V, \nabla_{\gamma'} \frac{\gamma'}{|\gamma'|} \rangle ds +$

$\langle V, \gamma' / |\gamma'| \rangle |_a^b$.

<u>Pf.</u> $\frac{d}{dt} \langle \varphi_s, \varphi_s \rangle^{\frac{1}{2}} = \frac{1}{2} \frac{1}{|\varphi_s|} \cdot \frac{d \langle \varphi_s, \varphi_s \rangle}{dt}$

$\Rightarrow d L(\gamma) / dt = \langle \varphi_t, \frac{\varphi_s}{|\varphi_s|} \rangle |_a^b -$

$\int_a^b \langle \varphi_t, \nabla_{\varphi_s} \frac{\varphi_s}{|\varphi_s|} \rangle ds$.

<u>Cor.</u> (For piecewise smooth γ)

If φ is variation of piecewise smooth γ on each $[s_i, s_{i+1}]$. and

$[a, b] = \bigcup_1^k [s_i, s_{i+1}]$. Then:

$\delta_V E(\gamma) = \langle V, \gamma' \rangle |_a^b - \int_a^b \langle V, \nabla_{\gamma'} \gamma' \rangle ds$

$- \sum_1^k \langle V(s_i), \gamma'(s_i^+) - \gamma'(s_i^-) \rangle$

$\delta_V L(\gamma) = \langle V, \gamma' / |\gamma'| \rangle |_a^b - \int_a^b \langle V, \nabla_{\gamma'} \frac{\gamma'}{|\gamma'|} \rangle ds$

$- \sum_1^k \langle V(s_i), \gamma'(s_i^+) / |\gamma'(s_i^+)| - \gamma'(s_i^-) / |\gamma'(s_i^-)| \rangle$

prop. γ is critical point of energy func.

(i.e. $\delta_V E(\gamma) = 0$) under $V_a = V_b = 0$ $\Leftrightarrow$

γ is geodesic.

Pf: By Thm above: $0 = -\int_a^b \langle V, \nabla_{\gamma'} \gamma' \rangle ds$

Cor. If γ minimizes $E(\gamma)$ or $L(\gamma)$. in C_{pq}

Then: γ is a geodesic.

Rmk: Stronger result: in each homotopy class, the length minimizing curve is geodesic.

Cor. For piecewise smooth curve γ:

γ is critical pt of energy func.

$\Rightarrow$ γ is C^1 and a geodesic.

Pf: We're freely to choose V:

i) Let $V(a) = V(b) = V_{s_i} = 0$. $\forall i$.

$\Rightarrow \nabla_{\gamma'} \gamma' = 0$. So γ is geodesic

2) Let $V(a) = V(b) = V_{s_j} = 0$. $\forall j \neq i$.

$\Rightarrow \gamma'(s_i^+) = \gamma'(s_i^-)$. So $\gamma \in C^1$

Thm: (Second derivative for variation)

$$\delta_V^2 E(\gamma) := \frac{d^2}{dt^2}\Big|_{t=0} E(\gamma_t)$$

$$= \langle \nabla_V \gamma_t, \gamma' \rangle \Big|_a^b - \int_a^b \langle \nabla_V \gamma_t, \nabla_{\gamma'} \gamma' \rangle ds$$

$$+ \int_a^b |\nabla_{\gamma'} V|^2 - \langle R(V, \gamma')\gamma', V \rangle ds$$

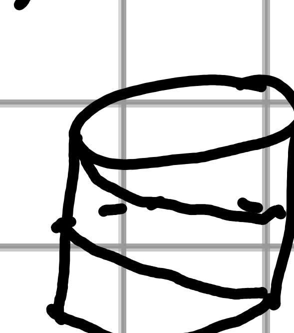
Pf: As 1^{st} variation formula. (Let $\partial/\partial s$ out by lem. to integrate)

Cor. $\gamma_0 \in C_{pq}$ is geodesic. Then: any sufficient small interval $[\tilde{a}, \tilde{b}] \subset [a, b]$, $\gamma_0|_{[\tilde{a}, \tilde{b}]}$ is locally minizing in $C_{\tilde{p}\tilde{q}}$.

Pf: By Wirtinger's inequi. We can let $\tilde{b} - \tilde{a}$ small enough to let $d^2E(\gamma)/dt^2 > 0$ under proper variation $V(a) = V(b) = 0$.

Rmk: geodesic may not minimize the length on the whole $[a, b]$.

e.g. helical geodesic around a cylinder.



Cor. If M has $K \le 0$, then for $\forall V \neq 0$, with fixed endpt $\Rightarrow \delta_V^2 E(\gamma) > 0$.

So any geodesic is locally minimizes length on $[a,b]$

Cor. For γ is geodesic and $V(a) = V(b) = 0$. We have:

$\delta_V^2 E(\gamma) = -\int_a^b \langle R(V,\gamma')\gamma' + \nabla_{\gamma'}\nabla_{\gamma'} V, V\rangle ds$

Pf: Integrate by part.

(2) Jacobi Fields:

Def: γ is geodesic in M, then the variation field V along γ is Jacobi field if

$R(V,\gamma')\gamma' + \nabla_{\gamma'}\nabla_{\gamma'} V = 0$

Rmk: i) We see above if $V_a = V_b = 0$, then:

$\delta_V^2 E(\gamma) = 0$

ii) γ is geodesic $\Rightarrow X = \gamma', s\gamma'(s)$ are Jacobi field, but not $s^2\gamma'$.

Since Jacobi field form a LS

$\Rightarrow X = (as+b)\gamma'(s)$ is Jacobi.

corspd to $\varphi_t(s) = \varphi(c(s+a)t)$

ii) A variation φ is called geodesic variation if $\forall t \in (-\varepsilon, \varepsilon)$. $\varphi(s,t) = \varphi_t$ is geodesic in M.

Thm. $\gamma : [a,b] \to M$ geodesic. $\Rightarrow \forall X_{\gamma(a)}, Y_{\gamma(a)} \in T_{\gamma(a)} M$. $\exists$ unique Jacobi field V along γ, st. $V_a = X_{\gamma(a)}$, $\nabla_{\gamma'(a)} V = Y_{\gamma(a)}$.

Pf: Assume $(e_i(s))_1^m \subset T_{\gamma(s)} M$ o.n.b. & parallel along γ. i.e. $\nabla_{\gamma'} e_k(s) = 0$

For $V = V^i(s) e_i(s)$ along γ. $V_s^i \in \mathbb{R}^1$.

$\Rightarrow \nabla_{\gamma'} \nabla_{\gamma'} V = \nabla_{\gamma'} (\dot{V}^i(s) e_i(s)) = \ddot{V}^i(s) e_i(s)$

So, Jacobi equation: (WLOG. $e_1(s) = \gamma'(s)$)

$X''^i(s) e_i(s) + X^i(s) R_{1i1}^{\ \ \ j} e_j(s) = 0$.

which forms a 2^{nd}-order ODE.

Cor. Set of Jacobi fields along γ is a LS of $\dim = 2m \underset{\text{isomorphic}}{\cong} T_{\gamma(a)} M \oplus T_{\gamma(a)} M$.

Cor. $V(s)$ is Jacobi field along γ. If $V \not\equiv 0$. then $Z(V)$ is discrete.

Pf: If $\exists (s_n) \to s_0$.

Set $V(s) = V^i(s) e_i(s)$, then:

$V^i(s_0) = 0$, $\dot{V}^i(s_0) = 0$. (Def of limit)

i.e. $V_{\gamma(s_0)} = 0$. $\nabla_{\gamma'} V_{\gamma(s_0)} = 0$.

$\overset{\text{uni}}{\Longrightarrow}$ $V \equiv 0$.

prop. Vector field V along γ is Jacobi field $\Leftrightarrow$ V is variation field of some geodesic variation of γ.

Pf: ($\Leftarrow$) $0 = \nabla_{\varphi_t}(\nabla_{\varphi_s} \varphi_s)$

$\overset{\text{lem}}{=} \nabla_{\varphi_s}(\nabla_{\varphi_s} \varphi_t) + R(\varphi_t, \varphi_s)\varphi_s$

($\Rightarrow$) Set $Y_{\gamma(a)} = \nabla_{\gamma'(a)} V$.

i) $V_{(a)} \neq 0$:

Let $\zeta : (-\varepsilon, \varepsilon) \to M$ is geodesic with $\zeta(0) = \gamma(a)$ & $\zeta'(0) = V_{\gamma(a)}$.

Let $T(s)$, $W(s)$ are parallel v.f. along ζ. s.t. $T(0) = \gamma'(a)$ and $W(0) = Y_{\gamma(a)}$

Set $\varphi : [a, b] \times (-\varepsilon, \varepsilon) \to M$

by $(s,t) \mapsto \exp_{\gamma(t)}((s-a)(T(t)+tW(t)))$

So φ is geodesic variation.

Next, check $\varphi_t(s,0) = V_s$

$\Leftrightarrow$ check $\varphi_t(a,0) = V_a$, $\nabla_{\gamma'(a)} \varphi_t(\cdot,0) = Y_{\gamma(a)}$

(using rule $\frac{d}{dt}\big|_{t=a}(\exp_p(g(t))) =$

$d(\exp_p)_{g(a)}\frac{d}{dt}\big|_{t=a} g(t))$

$\varphi_t(a,0) = \frac{d}{dt}\big|_{t=0} \varphi(a,t) = \frac{d}{dt}\big|_{t=0} \gamma(t)$

$$\varphi_s(a,t) = d(\exp_{\gamma(t)})_0 \frac{d}{ds}((s-a)(\underbrace{\Rightarrow}_{=id.}))$$

$= T(t) + tW(t).$

$\nabla_{\gamma'(a)} \square = \nabla_{\varphi_s} \varphi_t \big|_{\substack{s=a \\ t=0}} = \nabla_{\varphi_t} \varphi_s \big|_{s=a, t=0}$

$= \nabla_{\varphi_t}(T(t) + tW(t)) \big|_{t=0} = W(0)$

(since $\varphi_t(a,t) = \exp_{\gamma(t)}(0) = \gamma(t)$)

2) $V(a) = 0$:

Set $\varphi(s,t) = \exp_{\gamma(a)}((s-a)(\gamma'(a) + tY_{\gamma(a)}))$

Check as above!

<u>Rmk</u>: geodesic is def on open interval for def the deri. The lift. of this two case is that

whether $\gamma(a)$ is inner pt. of the image of $\varphi(s,t)$. When $V_{\gamma(a)} = 0$. $\Rightarrow$ it's no longer inner pt.!

$\Rightarrow$ ξ can't be defined!

Df: A Jacobi field along γ is normal Jacobi field if $V \perp \gamma'(s)$ along γ.

Prop: V is Jacobi field along $\gamma \Rightarrow \exists\, c, k \in \mathbb{R}'$ st. $V^{\perp} \triangleq V - cs\gamma'(s) - k\gamma'(s)$ is normal Jacobi field. along γ.

Pf: 1) $V^{\perp}$ is Jacobi field (linear comb.)

2) $\frac{d^2}{ds^2} \langle V, \gamma' \rangle = \langle \nabla_{\gamma'} \nabla_{\gamma'} V, \gamma' \rangle$

$= -\langle R(\gamma', V)\gamma', \gamma' \rangle = 0$

$\Rightarrow \langle V, \gamma' \rangle = \tilde{c}s + \tilde{k}$, $\exists\, \tilde{c}, \tilde{k} \in \mathbb{R}'$

Let $c = \tilde{c}/|\gamma'|^2$, $k = \tilde{k}/|\gamma'|^2$.

Cor. Jacobi field V is normal $\Leftrightarrow$

$\langle V(a), \gamma'(a) \rangle = \langle \nabla_{\gamma'(a)} V, \gamma'(a) \rangle = 0$

Pf: $V = V^{\perp} + cs\gamma'(s) + k\gamma'(s)$

So: $\langle V(a), \gamma'(a) \rangle = (ca + k)|\gamma'|^2$

$$\langle \nabla_{\gamma'(a)} V, \gamma'(a) \rangle = c|\gamma'|^2$$

Rmk: So we have set of normal Jacobi field is LS of dim $= 2m-2$

Cor. V is Jacobi field s.t. $\exists s_1 \neq s_2$

$\langle V(s_1), \gamma'(s_1) \rangle = \langle V(s_2), \gamma'(s_2) \rangle = 0$

Then: V is normal Jacobi.

Pf: $\langle V, \gamma' \rangle = ct + d$ is LF.

Eg. (M, g) is R-M has const k. Then:

$$R(X, Y)Z = k(\langle X, Z \rangle Y - \langle Y, Z \rangle X)$$

$\Rightarrow R(V, \gamma')\gamma' = kV$. if $|\gamma'|^2 = 1$, $\langle V, \gamma' \rangle = 0$

i.e. $\nabla_{\gamma'} \nabla_{\gamma'} X + kX = 0$ is the equation of a normal Jacobi field X.

As we did before, we have:

$k\ddot{X}^i(t) + kX^i(t) = 0$, $\forall\, 1 \le i \le m \Rightarrow$ solve X^i.

(3) Conjugate pt:

Def: (M, g) is R-M, $\gamma: [a, b] \to M$ geodesic

$q = \gamma(t_0)$, $t_0 > a$ is conjugate to $p = \gamma(a)$ if $\exp_p$ is singular at $(t_0-a)\dot\gamma(a)$

i.e. $(d\exp_p)_{(t_0-a)\dot\gamma(a)}$ isn't full rank.

Rmk: i) $\exp_{\gamma(a)}((t-a)\dot\gamma(a)) = \gamma(t)$, corrspd to geodesic $\gamma((t-a)s+a)$

ii) Recall $\exp_p$ is diffeo near 0. But it may fail away from 0.

Thm. γ is geodesic from $[a,b]$ to M.

Then: $q = \gamma(t_0)$ is conjugate to $p = \gamma(a)$ $\Longleftrightarrow$ $\exists$ Jacobi field V along γ. $V \not\equiv 0$. st. $V(a) = V(t_0) = 0$

Pf: $\Longleftrightarrow$ $\ker (d\exp_p)_{(t_0-a)\dot\gamma(a)} \neq 0$

$\Longleftrightarrow$ $\exists\ Y_{\gamma(a)} \neq 0$. st.

$0 = (d\exp_{\gamma(a)})_{(t_0-a)\dot\gamma(a)}((t_0-a)Y_{\gamma(a)})$

Since by prop. in (2), we have Jacobi field V along γ. st. $V(a) = 0$. $\nabla_{\dot\gamma(a)} V = Y_{\gamma(a)}$. with form:

$\varphi(s,t) = \exp_{\gamma(a)}((s-a)(\gamma'(a) + tY_{\gamma(a)}))$

So: $V(t_0) = (d\exp_{\gamma(a)})_{(t_0-a)\gamma'(a)}((t_0-a)Y_{(a)}) = 0$

<u>Rmk</u>: We also see dim of {Jacobi field vanishing on p, q} = dim $(\ker (d\exp_p)_{(t_0-a)\gamma'(a)}) \le m-1$ = dim {normal Jacobi field with vanishing on p} from last cor. in (2).

<u>e.g.</u> On the last example in (2). We set $M = S^m$. $\Rightarrow K_M = 1$. So normal Jacobi field is $V(t) = \sum^m c_k \sin(t) e_k(t)$.

i) if $t < \pi$. (i.e. $L(\gamma) < \pi$. assume $|\gamma'| = 1$) then there's no conjugate pt.

ii) if $L(\gamma) \in (\pi, 2\pi)$, then the antipodal pt $\gamma(\pi) = \overline{\gamma(0)}$ is the only conjugate pt. And its multiplicity $= m-1$.

<u>Rmk</u>: As above. for M with const. $k \le 0$. There's no conjugate pt.

Prop. $q = \gamma(t_0)$ isn't conjugate to $p = \gamma(a)$.

$\Rightarrow \forall X_p \in T_pM.\ X_q \in T_qM.$ there exists unique Jacobi field along geodesic γ s.t. $V(a) = X_p.\ V(t_0) = X_q.$

<u>Pf</u>: $\Theta : \mathcal{J} \longrightarrow T_pM \times T_qM$

$$V \longmapsto (V(a), V(t_0))$$

$\mathcal{J}$ = set of Jacobi field along γ

$\Rightarrow \Theta$ is linear isomorphism.

Thm. (Global length-min among nearby)

$\gamma : [a,b] \to M$. geodesic. $p = \gamma(a)$. $q = \gamma(b)$

If there's no conjugate pt of p along γ. Then $\exists\ \varepsilon > 0$. s.t. $\forall\ \tilde{\gamma} \in C_{pq}$. satisfies: $d(\gamma, \tilde{\gamma}) < \varepsilon. \Rightarrow L(\tilde{\gamma}) \geqslant L(\gamma).$

<u>Rmk</u>: i) Conversely, if $\exists$ conjugate pt on γ. then any curve contain γ won't be length-minizing

ii) It's consistent with case $k \leq 0$. $k \equiv$ const. $\Rightarrow \delta_V^2 E(\gamma) > 0$.