Tangent Space II

(1) Tangent Space

T_pM is collection of velocity vectors to curves through p.

Consider $M = \mathbb{R}^n$. $g : \mathbb{R}^n \to \mathbb{R}^1$. γ is curve passing p at $t=0$. $v = \gamma'(0) \in T_p\mathbb{R}^n$ is its velocity vector.

$\Rightarrow \partial_v g = D_p g(v) = \frac{d}{dt}\Big|_{t=0}(g \circ \gamma) = g'(p) \cdot v$

We want to see tangent vector as directional derivative on $f : M \to \mathbb{R}^1$.

Def: Fix $p \in M^m$, m-dim manifold.

i) $g : U_p \to \mathbb{R}^1$ is equi- with $f : V_p \to \mathbb{R}^1$, where U_p, V_p are nbhds of p, if $\exists$ nbd $W_p \subset U_p \cap V_p$ s.t. $g|_{W_p} = f|_{W_p}$.

ii) $C = \bigcup_{\substack{U \ni p \\ \text{nbd}}} C^\infty(U)$. We denote: the equi. relation in i) is $\sim$

Set $C^\infty(p) = C/\sim$. germ at p.

iii) $T_pM = \{ X_p : C^\infty(p) \to \mathbb{R}^1 .\ LF \mid X_p(g + \lambda h) = X_p g + \lambda X_p h,\ X_p(gh) = (X_p g) h(p) + (X_p h)\cdot g(p),\ \lambda \in \mathbb{R}^1 \}$.

Rmk: i) $T_pM \subsetneq C^\infty(p)^*$.

ii) Note $T_pU = T_pM$. if $p \in U \subset M$

Consider $f : M \to N$. Smooth map of mfds.

induce $f^* : C^\infty(f(q)) \to C^\infty(q)$, s.t. $f^*(g) = g \circ f$.

$g \mapsto f^*(g)$

induce $f_* : T_pM \to T_{f(p)}N$ $\quad f_*(X_p)(g) = X_p(f^*(g))$

$X_p \mapsto f_*(X_p)$ $\quad \forall g \in C^\infty f(p)$

Thm. Under conditions above. $f_* \overset{\Delta}{=} D_p f : T_p M \to T_{f(p)} N$ is linear mapping.

Pf: Check: $f_*(X_p) \in T_{f(p)} N$.

linear is trivial. So we check Leibniz:

$f_*(X_p)(gh) = X_p((g\circ f)(h\circ f)) = \square$

Cor. $D_{f_1(p)} f_2 \circ D_p f_1 = D_p(f_2 \circ f_1)$

Pf: Check $D_{f_1(p)} f_2 \circ D_p f_1(X_p)(g)$

$\overset{\forall g}{=} D_p(f_2 \circ f_1)(X_p)(g)$.

Cor. $f: M \to N$. diffeomorphism $\Rightarrow$

$\forall p \in M.\ D_p f : T_p M \xrightarrow{\sim} T_{f(p)} N$. isomor.

Pf: $D_p(f^{-1}\circ f) = id_* = D_{f(p)} f^{-1} \circ D_p f$

$\Rightarrow D_p f$ is isomor.

Next, we want to prove $\mathbb{R}^m \simeq T_p \mathbb{R}^m$. $\forall p \in \mathbb{R}^m$.

Set $z: \mathbb{R}^m \to \mathbb{R}^1$. $p \mapsto p^i$. $\Rightarrow \partial_v z^i = v^i = z^i(v)$.

Lemma. $X_p \in T_p \mathbb{R}^m$. $g \in C^\infty(p)$ is const. $\Rightarrow$

$X_p(g) = 0$.

Pf: WLOG. $g \equiv 1$. $\Rightarrow X_p(1) = X_p(1 \cdot 1)$

$\overset{\text{Lebniz}}{=} X_p(1) \cdot 1 + X_p(1) \cdot 1 \Rightarrow X_p(1) = 0$.

Lemma^2. $B(p, \varepsilon) \overset{\Delta}{=} B \subset \mathbb{R}^n$. $\forall g \in C^\infty(B)$. $\exists (h_i)_1^n \in C^\infty(B)$. s.t. $h_i(p) = \frac{\partial g}{\partial x^i}(p)$. and

$g(x) = g(p) + \sum_1^n (x^i - p^i) h_i(x)$ on B.

Pf: $g(x) - g(p) = \int_0^1 \frac{d}{dt} g(p + t(x-p)) dt$

$\Rightarrow h_i(x) = \int_0^1 \frac{\partial g}{\partial x^i}(p + t(x-p)) dt$

Thm. $\varphi : \mathbb{R}^n \to T_p\mathbb{R}^n$. is isomorphism

$v \mapsto \partial_v$

Pf: 1) $\partial_v(z_i) = \partial_w(z_i) \Leftrightarrow \forall i.\ v_i = w_i$

$\Rightarrow \varphi$ is injective

2) $\forall X_p \in T_p\mathbb{R}^n$. we set $v \in \mathbb{R}^n$ by

$v^i = X_p(z^i)$. For $\forall g \in C^\infty(B)$.

$X_p g \overset{\text{Lem}^2}{\underset{\text{Leb}}{=}} X_p(g(p)) + \sum (X_p z^i - X_p(p_i))(h_i(p))$

$+ \sum X_p(h_i)(z^i(p) - p_i)$

$\overset{\text{Lem}^1}{=} \sum X_p z^i \cdot h_i(p)$

$\overset{\text{Lem}^2}{=} \sum v^i \cdot \frac{\partial g}{\partial x^i}(p) = \partial_v g$.

Rmk: i) If $(e_i)_1^m$ is basis of $\mathbb{R}^m$. Then we

set $\{\partial e_i = \frac{\partial}{\partial x_i}\}_1^m$ is basis of $T_p\mathbb{R}^m$

Note if given local chart (U, φ)

on $p \in M^m$. Since $\varphi_* : T_pM \xrightarrow{\sim} T_{\varphi(p)}\mathbb{R}^m$

We can set $(\partial_{i.p}) := (\varphi_*^{-1}(\frac{\partial}{\partial x_i}))$

is basis of T_pM.

since $\varphi_* X_p = \sum_1^m \overbrace{\varphi_* X_p(x_i)}^{= \text{const.}} \cdot \frac{\partial}{\partial x^i}$

$\Rightarrow X_p = \sum_1^m X_p(x_i \circ \varphi)\ \varphi_*^{-1}(\frac{\partial}{\partial x^i})$

$= \sum_1^m X^i \cdot \partial_{i.p}$. expressed in basis.

ii) Consider $f : M^m \to N^n$. smooth. And

(U, φ), (V, ψ) is local chart on

$p \in M$ and $f(p) \in V$.

Consider $\psi \circ f \circ \varphi^{-1} : (x_1 \cdots x_m) \mapsto (y_1 \cdots y_n)$

where $y^i = f(x_1 \cdots x_m)$

$\Rightarrow$ Its derivative is $J = (\frac{\partial y^i}{\partial x_j})$.

Denote $\tilde{\partial}_{j.q} = \psi_*^{-1}(\partial/\partial y^j)$.

$\Rightarrow D_pf(\partial_{i.p}) = \psi_*^{-1}(\psi_* \circ f_* \circ \varphi_*^{-1}(\partial/\partial x_i))$

$= \psi_*^{-1}\left(\sum_j \frac{\partial y_j}{\partial x_i}\Big|_{\varphi(p)} \partial/\partial y_j \right) = \sum_j \frac{\partial y_j}{\partial x_i}\Big|_{\varphi(p)} \tilde{\partial}_{j,e}.$

Note if set $f = id$. (U, φ), (V, ψ) are charts on M. correspond to 2 bases. $\Rightarrow$ transition matrix of the 2 bases is just $J = \psi_* \circ (\varphi_*)^{-1}$

Def. $f: M^m \to N^n$ has rank $= k$ at $p \in M$. if $D_p f : T_p M \to T_{f(p)} N$ has rank $= k$.

Thm. (Rank Thm).

$f: M^m \to N^n$. smooth have local const. rank $= k$. $\Rightarrow$ $\forall p \in M$. $\exists$ charts (U, φ) and (V, ψ) on M. N. s.t.

$\psi \circ f \circ \varphi^{-1} : (x^1 \cdots x^m) \mapsto (x^1 \cdots x^k, 0 \cdots 0)$.

Rmk: f has rank $m \le n$. $\Rightarrow$ Immersion

f has rank $n \le m$ $\Rightarrow$ Submersion.

Def: i) $\gamma : (a,b) \to M^m$. curve. $p = \gamma(t)$.

Set $\gamma'(t) \stackrel{\Delta}{=} D_t\gamma(\partial t) \in T_pM$.

ii) $f : M^m \to \mathbb{R}^1$. locally on (U, φ):

$$D_pf(X)(g) = (\sum_i X^i \partial_i)(g \circ f)$$

$$= \sum_i X^i \varphi_*^{-1}(\frac{\partial}{\partial x_i})(g \circ f) = \sum X^i \partial_t g \frac{\partial}{\partial x_i} f \circ \varphi^{-1}$$

$$= \sum X^i \partial_i f \cdot \partial_t g = Xf \cdot \partial_t g.$$

Set $d_pf : T_pM \to \mathbb{R}^1$, $X \mapsto Xf$, i.e. directional derivative on X.

So: $D_pf(X) = d_pf(X) \cdot \partial_t$.

(2) <u>Tangent Bundle</u>:

Set $TM^m = \bigcup_{p \in M} T_pM$. with projection

$\pi : TM \to M$. so. $\pi^{-1}[p] = T_pM$.

prop. We can equip TM with $2m$ dim smooth manifold structure.

Pf: $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is local chart of M. We have:

$$D\varphi_\alpha : TU \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^m \subseteq \mathbb{R}^{2m}$$

$$\underset{\substack{\cap \\ T_pU = T_pM}}{\sum v^i \partial_i} \mapsto (\varphi_\alpha(p), v)$$

equip TM with topology:

$\{D\varphi_\alpha^{-1}(V) \mid V \underset{\text{open}}{\subseteq} \varphi_\alpha(U_\alpha) \times \mathbb{R}^m, \alpha \in A\}$

$\Rightarrow D\varphi_\alpha$ is homeomorphism.

Since M is C_2. We can choose A is countable index set. $\Rightarrow$ It form a countable basis of TM.

Hausdorff of TM follows from $D\varphi_\alpha$ is homeomorphic.

$\{(D\varphi_\alpha, TU_\alpha)\}$ is also compatible

Prop. $T(M^m \times N^n) \underset{\text{isomor.}}{\cong} TM^m \oplus TN^n$.

Pf: $M\times N \xrightarrow{z_1} M$. $M\times N \xrightarrow{z_2} N$.

See $T_{(p,q)} M\times N \xrightarrow{f} T_pM \times T_qN$

$z \mapsto (D_{(p,q)}z_1(z), D_{(p,q)}z_2(z))$

has inverse:

$M \xrightarrow{S_1} M\times N$ $\quad$ $N \xrightarrow{S_2} M\times N$

$x \mapsto (x,q)$ $\quad$ $y \mapsto (p,y)$.

$T_pM\times T_qN \xrightarrow{g} T_{(p,q)} M\times N$.

$(X, Y) \mapsto D_pS_1(X) + D_qS_2(Y)$

$\Rightarrow f\circ g = id$. (with same dim.)

So: $T_{(p,q)} M\times N \cong T_pM \oplus T_qN$.

$\Rightarrow TM\times N \overset{Dz_1\times Dz_2}{\cong} TM \oplus TN$.

Rmk: We can write $T(M\times N)$ into $\{(x,y,v,w) \in M\times N\times \mathbb{R}^m\times\mathbb{R}^n\}$ and also get canonical diffeomorphic.

Def: Vector field X on manifold M is defined by $X: p \in M \mapsto X_p \in T_pM$.

$\mathfrak{X}(M)$ is set of vector field on M.

Rmk: i) $\pi \circ X = id_M$.

ii) Define addition $(X+Y)_p = X_p + Y_p$.

prop: $X \in \mathfrak{X}(M)$. Then: X is smooth v.f.

$\overset{i)}{\iff} \forall f \in C^\infty(M)$, $Xf(p) := X_pf$ is smooth

$\overset{ii)}{\iff}$ On local chart $X|_U = \sum v^i \partial_i$ with $v^i \in C^\infty$

Pf: (U, φ). $(TU, D\varphi)$ is local chart

on M, TM. For ii):

$D\varphi \circ X \circ \varphi^{-1} = (\varphi \circ \varphi^{-1}, v^1 \circ \varphi^{-1} \cdots v^m \circ \varphi^{-1})$

For i): we only need to consider

in local chart (U, φ). $f \in C^\infty(U)$.

$X_pf = \sum v^i(p) \partial_{i,p} f$ is smooth

$\iff v^i(p): U \to \mathbb{R}$ is smooth.

(choose $f = x_i$. $\Rightarrow v_i \in C^\infty$)