Vector Flows

(1) Definitions:

Def: i) $Y \in \mathfrak{X}(N)$ is f-related to $X \in \mathfrak{X}(M)$

if $Y_{f(p)} = D_p f(X_p)$. $\forall p \in M$.

Rmk: If f isn't injective, $Y_{f(p)} = Y_{f(p')}$ might happen.

If f isn't surjective, then $Y_{f(p)} \in \mathfrak{X}(N)$ can't be uniquely defined when $q \notin f(M)$

When f is diffeo. $\Rightarrow$ it's well-def.

ii) For $f: M \to M$. diffeomorphism. We say X is f-related to itself. i.e. f-invariant. if $X = f_*(X)$.

Def: i) G is algebraic group. and X is a set. G-action θ on set X is map:

$\theta: G \times X \to X$. $(g, x) \mapsto g \cdot x \overset{\triangle}{=} \theta_g(x)$

satisfying $\theta_e = id_X$. $\theta_{gh} = \theta_g \circ \theta_h$.

Rmk: i) Each θ_g is bijection with inverse $\theta_{g^{-1}}$

ii) Orbit $G \cdot X \stackrel{\Delta}{=} \{ g \cdot x \mid g \in G \}$.

Next, we set $G = (\mathbb{R}^{\geq 0}, +)$. $X = M^n$. θ_t is smooth map (so diffeomorphic) called flow on M.

Def: i) $X \in \mathfrak{X}(M)$ is θ-invariant if $(\theta_t)_* X = X$. $\forall t$

Rmk: Fix pt. $p \in M$. Set $\gamma_p(t) = \theta_t(p)$. Note if $q = \gamma_p(s)$ then $\gamma_q(t) = \gamma_p(t+s)$.

ii) Infestimal generator of flow θ is $X \in \mathfrak{X}(M)$ defined by:

$X_p = \gamma_p'(0)$. velocity vector of γ_p.

Rmk: Equivalently. $X_p = (D_{(0,p)}\theta)(V)$.

$V_{(t,p)} = (\partial t, 0) \in T_{(t,p)}(\mathbb{R}^1 \times M)$.

Thm: θ is flow on M of infinitesimal generator X. $\Rightarrow X$ is θ-invariant.

i.e. $(\theta_s)_* X_p = X_{\theta_s(p)}$.

$(X_{\theta(t,p)} = D_{(t,p)}\theta\,(U_{(t,p)})$

Pf: $(\theta_s)_* X_p = D_p\theta_s \circ D_0 \gamma_p(\partial t)$

$= d(\theta_s \circ \gamma_p(t))_0(\partial t)$

$= d(\theta_t(\theta_s(p)))_0(\partial t) = X_{\theta_s(p)}$

Cor. For $X_p \equiv 0$. $\Rightarrow$ $\gamma_p(t) \equiv p$. const.

For $X_p \neq 0$. $\Rightarrow$ γ_p is immersion

if it's not injective additional.

$\Rightarrow$ It's s-periodic. injective on $\mathbb{Z}/s\mathbb{Z}$

Pf: $\gamma_p'(t) = \gamma_{\gamma_p(t)}'(0) = X_{\gamma_p(t)}$ $(*)$

$\overset{\text{Thm}}{=} (\theta_t)_* X_p$

Since θ_t is diffeo. So $(\theta_t)_*$ is isomorphic.

if $X_p \neq 0$. then γ_p' is injective

Rmk: By flow prop. and $(*)$. generator (X_p) determines $\forall$ $\gamma_p'(t)$.

i.e. φ_p is a immersion

Note immersion is locally injective

if $\exists\, t.\ s>0$. s.t. $\varphi_p(t+s) = \varphi_p(t)$

then it holds for $\forall t>0$

(2) <u>Infestimal generator:</u>

Start with vector field $X \in \mathfrak{X}(M)$.

Does it generate a flow?

<u>Def</u>: $X \in \mathfrak{X}(M)$. vector field. $J \subseteq \mathbb{R}^1$ open interval. A curve $\gamma: J \to M$ is integral curve of X. if we have:

$\gamma'(t) = X_{\gamma(t)}$. $\forall t \in J$.

<u>Rmk</u>: i) It means gradient of $\gamma(t)$ will follows vector field X_p. at each $p = \gamma(t)$. And $\gamma(t) = \int_\gamma X_p\, dp$

ii) The integral curve may not exist all the time. since it may flow out of M.

(e.g. $\theta_t(x) = x + te_1$. $X = \partial_1$. $M = B_1(0)$)

__Thm__: $U \underset{\text{open}}{\subseteq} \mathbb{R}^m$. $f: U \to \mathbb{R}^m$. smooth. Then:

$\forall p \in U$. $\exists$ unique solution for $\frac{dx}{dt} = f(x)$

with $x(0) = p$. which's smooth and def

on some maximal interval $(a_p, b_p) \ni 0$.

__Pf__: By Banach fixed pt. Thm.

__Thm.__ Under conditions above. $\Rightarrow \forall p \in U$. $\exists \varepsilon > 0$

nbd $V \subset U$ of p. s.t. exists unique C^∞

map $x: (-\varepsilon, \varepsilon) \times V \to U$. Satisfies:

$\frac{\partial x}{\partial t}(t, q) = f(x(t, q))$. $x(0, q) = q$.

for $\forall t \in (-\varepsilon, \varepsilon)$. $q \in V$.

__Def.__ A local flow around $p \in M$ is map

$\theta: (-\varepsilon, \varepsilon) \times V \to M$. for some $\varepsilon > 0$ and

$p \in V \underset{\text{open}}{\subseteq} M$. s.t. $\theta_0(q) = q$. $\forall q \in V$. and

$\theta_t(\theta_s(q)) = \theta_{t+s}(q)$.

The flow line is $\gamma_p(t) = \theta_t(p)$. with

infinitesimal generator $X_q = \gamma_q'(0) \in \mathfrak{X}(M)$.

Rmk: Note $X_{\theta_t(q)} = X_{\varphi_q(t)} = \varphi_q'(t)$

$\Rightarrow$ it's also a integral curve for its generator X_p.

Thm. Any $X \in \mathfrak{X}(M)$. has unique local flow $\varphi_p(t)$ around $p \in M$. (i.e. σ_p is another s.t. $\sigma_p(0) = p$. $\Rightarrow \varphi_p = \sigma_p$.)

Pf: By previous Thm on $M = \mathbb{R}^n$. We can choose charts to prove (~~Set~~ $f = X$. . Solve $X_t = \varphi_t$.) Uniqueness is from Lem. below

Lemma. For local flows $\alpha(t)$, $\beta(t)$ on $(-\varepsilon_1, \varepsilon_1) \times U_p$, $(-\varepsilon_2, \varepsilon_2) \times \widetilde{U_p}$, $\alpha_r(0) = \beta_r(0)$, $\forall r \in J = U_p \cap \widetilde{U_p}$ $\Rightarrow \alpha = \beta$ on $T \times J$. $T = (-\varepsilon_1, \varepsilon_1) \cap (-\varepsilon_2, \varepsilon_2)$

Pf: Fix $r \in J$. $S = \{ t \in T : \alpha_r(t) = \beta_r(t) \}$.

$0 \in S$. And $\forall t \in S$, By Thm on uniqueness of ODE. above. $\exists\, \mathcal{U}$ nbd. of $\alpha_r(t) = \beta_r(t)$. s.t. they coincide with each other on $\mathcal{U}$.

$\Rightarrow$ S is open.

And mt for $(\alpha, \beta) : T \to M \times M$.

$S = (\alpha, \beta)^{-1}(\Delta)$. Δ is diagonal of $M \times M$

$\Rightarrow$ S is closed (M is Hausdorff)

So: $S = T$ by connected

<u>Cor.</u> On cpt mfd M^n. $\forall X \in \mathfrak{X}(M)$

field has a global flow.

<u>Pf</u>: $\forall p \in M$. $\exists$ local flow defined on $(-\varepsilon_p, \varepsilon_p) \times V_p$. From cptness: $\exists$ finite cover $(V_{p_i})_1^n$. for M.

set $\varepsilon = \min \varepsilon_p$ and $\theta_p(t) = \theta_p^i(t)$ if $p \in V_{p_i}$. $\forall t \in (-\varepsilon, \varepsilon)$. which's well-def by lem. above. Then extend θ_t by $\theta_{nt} = (\theta_t)^n$ on $\mathbb{R}'$.

<u>Cor.</u> $f : M \xrightarrow{\text{smooth}} N$. γ is integral curve for X. Then: Y is f-related to X

$\Longleftrightarrow$ $f \circ \gamma$ is integral curve of Y.