Lie Derivative

(1) Lie bracket :

Def: Lie algebra is v.s. $\mathcal{L}$ with a anti-symmetric product $[\cdot,\cdot] : \mathcal{L}\times\mathcal{L} \to \mathcal{L}$ is is bilinear. and satisfies Jacobi. id

$$\sum_{cyc} [u,[v,w]] = 0.$$

Rmk : i) Trivial $\mathcal{L}$-algebra : $[v,w] := 0$.

ii) Jacobi. id. can be viewed as a replacement of associaty.

e.g. $End(V) := L(V,V)$. linear endomorphism on V with $[A,B] \triangleq A\circ B - B\circ A$.

Recall $\mathfrak{X}(M)$ can be viewed as a module on $C^\infty(M)$: $(fX+Y)_p \triangleq f(p)X_p + Y_p$.

And recall $X \in \mathfrak{X}$: $\begin{aligned} C^\infty(M) &\to C^\infty(M). \\ f(p) &\mapsto (X_p f(p))_p. \end{aligned}$

So : $\mathfrak{X}(M) \subset End(C^\infty(M)) = ($ Aut $\cap$ Isomor. $)$

Thm. $\mathfrak{X}(M)$ is Lie algebra with $[X,Y]_p f$
$:= X_p(Yf) - Y_p(Xf)$. $\forall f \in C^\infty(M)$.

Rmk: i) $f \mapsto X(Yf)$ is endomorphism of
$C^\infty(M)$ should be thought as:
taking second derivative in some
direction. But it's not a v.f.
It doesn't satisfies Lebniz:
Consider on $M = \mathbb{R}^n$:

$$\frac{d^2}{ds^2}\Big|_{t=0} f(\gamma(t)) = f''(\gamma(0))(\gamma'(0))^2 + f'(\gamma(0))(\gamma''(0))$$

ii) $[X,Y]$ is $\mathbb{R}$ - bilinear. But not
$C^\infty(M)$ - bilinear

prop. $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$

Pf: check: $[fX, Y] = f[X,Y] - (Yf)X$.
let $Y := gY$.

Pf: i) Check it only locally depends on
$C^\infty(p)$.

2) Check it satisfies Leibniz rule.

<u>Prop.</u> $f: M^m \to N^n$. smooth. $X^i \in \mathfrak{X}(M)$. $Y^i \in \mathfrak{X}(N)$

X^i is f-related to Y^i. $i=1,2$. Then:

$[X^1, X^2]$ is f-related to $[Y^1, Y^2]$

<u>Pf:</u> $[Y^1, Y^2]_{f(p)} g = Y^1_{f(p)}(Y^2 g) - Y^2_{f(p)}(\square)$

$= D_p f(X^1_p)(Y^2 g) - D_p f(X^2_p)(\square)$

$= X^1_p(Y^2_{f(\cdot)} g) - X^2_p(Y^1_{f(\cdot)} g)$

$= [X^1, X^2]_p(g \circ f)$

<u>Cor.</u> $\varphi: M \to N$ is diffeomorphism.

$X^1, X^2 \in \mathfrak{X}(M)$. $\Rightarrow \varphi_*([X^1, X^2]_p)$

$= [\varphi_* X^1, \varphi_* X^2]_{\varphi(p)}$

<u>Pf:</u> Set $Y^i = \varphi_* X^i$. above.

<u>Cor.</u> (U, φ) is local chart. $(\partial_i) \in$

$\mathfrak{X}(M)$ is coordinate basis. $\Rightarrow$

$[\partial_i, \partial_j] = 0$. $\forall i, j$.

<u>Pf:</u> $LHS = \varphi_*^{-1}([\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]) \overset{\text{iso.}}{=} 0$

(2) <u>Lie deri.</u>:

1) Note $X.f$ is directional derivation of f along the flow line of X:

$$X_p f \overset{\gamma_p'(0)=X_p}{=} \frac{d}{dt}\Big|_{t=0} f(\gamma_p(t)) = \frac{d}{dt}\Big|_{t=0} f(\theta_t(p))$$

$$\overset{t=t+s}{\Rightarrow} X_{\theta_s(p)} f = \frac{d}{dt}\Big|_{t=s} f(\theta_t(p))$$

2) Next, if we want to derivate vector field Y rather $f \in C^\infty(M)$ along Y.

We need to overcome the problem: Y_q lives in different tangent space $T_q M$. So it's impossible to ask rate of change of Y_q along Y.

Def: $X, Y \in \mathfrak{X}(M)$. Lie derivative $L_X Y$ of Y w.r.t X is defined by

$$(L_X Y)_p := \frac{d}{dt}\Big|_{t=0} (\theta_{-t})_* Y_{\theta_t p} \in T_p M.$$

where θ_t is local flow of X.

Thm. $X, Y \in \mathfrak{X}(M) \Rightarrow L_X Y = [X, Y]$.

Pf: Lemma. $X \in \mathfrak{X}(M)$ with local flow θ : $(-\varepsilon, \varepsilon) \times U \longrightarrow M \ni p$. Then:

$\forall f \in C^\infty(M)$, $\exists$ smooth $g : (-\varepsilon, \varepsilon) \times U \to \mathbb{R}$, s.t. $X_q f = g_0(q)$. And that:

$f(\theta_t(q)) = f(q) + t\, g_t(q)$.

Pf: Let $h_t(q) := d f(\theta_t(q))/dt = df(\theta_t(q)')$

$= df(X_{\theta_t q}) = X_{\theta_t q} f$.

And let $g_t(q) = \int_0^1 h_{st}(q)\, ds$.

For $f \in C^\infty(p)$, by Lemma $\exists g$, s.t.

$f \circ \theta_{\pm t} = f \pm t\, g_{\pm t}$, $g_0 = Xf$.

$\Rightarrow (L_X Y)_p f = \lim_{t \to 0} \frac{1}{t} \left((\theta_{-t*} Y_{\theta_t p}) f - Y_p f \right)$

$= \frac{d}{dt}\Big|_{t=0} (Y_{\theta_t p} f) - \lim_{t \to 0} Y_{\theta_t p}\, g_{-t}$

$= \frac{d}{dt}\Big|_{t=0} (Yf)(\theta_t p) - Y_p\, g_0$

$= X_p(Yf) - Y_p(Xf)$.