Vector Bundles. One forms.

(1) Vector bundles:

Def: i) A bundle over base space M consists of total space E and projection $\pi : E \to M$.

ii) The fiber over $p \in M$ is $E_p = \pi^{-1}\{p\}$.

iii) We write E_S is restrict of E on S: $\pi|_S : E_S = \pi^{-1}(S) \to M$.

iv) $\pi' : E' \to M$ is another bundle. $\varphi : E \to E'$ is fiber-preserving if $\pi' \circ \varphi = \pi$. $\varphi(E_p) \subset E'_p$. $\forall p$.

Rmk: $\pi : TM \to M$ is a bundle where (U, φ) is trivializing nbh.

Def: A vector bundle of rank k is map $\pi : E^{n+k} \to M^n$. of mflds. st.

i) $\forall$ fiber $\pi^{-1}(\{p\})$, is real v.s. of dimension k.

ii) $\forall p \in M$ has a trivializing nbd U. i.e. $\exists$ fiber-presv. diffeomorphism $\varphi: \pi^{-1}(U) = E(U) \xrightarrow{\sim} U \times \mathbb{R}^k$. s.t. φ is v.s. isomorphism on each E_p.

$\underline{\text{eg.}}$ i) $M \times \mathbb{R}^k \xrightarrow{\pi} M$ is a trivial bundle.

ii) $TM \xrightarrow{\pi} M^m$ is vector bundle of $r = m$, $\varphi = (\pi, D\varphi) : X_p \mapsto (p, D_p\varphi(X_p))$

$\varphi|_{\pi^{-1}(U)} : X_p \xrightarrow{\sim} (p, D_p\varphi(X_p)) \in U \times \mathbb{R}^m$.

is diffeo. and $\varphi|_{\pi^{-1}(p)}$ is v.s. isomor.

$\underline{\text{Rmk}}$: (U, f) is chart on M and $\varphi: E_U \to U \times \mathbb{R}^k$ is fiber preserving diff. from Def. Then we also have local chart. on E. By considering:

$(f \cdot id) \circ \varphi : E_U \xrightarrow{\sim} f(U) \times \mathbb{R}^k \subseteq \mathbb{R}^{m+k}$

Def: A section of vector bundle $\pi: E \to M$ is σ. s.t. $\pi \circ \sigma = id_M$.

Denote set of such section on E by $\Gamma(E)$.

Rmk: i) $\Gamma(TM) \overset{\Delta}{=} \mathcal{X}(M)$.

ii) $\Gamma(E)$ is a module on $C^\infty(M)$:

$(f\sigma + \tau)_p = f(p)\sigma_p + \tau_p \in E_p$.

iii) Consider locally on (U, φ):

trivialization of E over U is equi. to frame. i.e. set of k sections $\sigma_i \in \Gamma(E_U)$. s.t. $\forall p \in U$. $(\sigma_i(p))$ is bases of E_p.

$(\Rightarrow)$ $\varphi: E_U \to U \times \mathbb{R}^k$. Let $\sigma_i(p) := \varphi^{-1}(p, e_i)$. (e_i) is basis of $\mathbb{R}^k$.

$(\Leftarrow)$. Set $\varphi: E_U \to U \times \mathbb{R}^k$ is trivialization $\sum_1^k \lambda_i \sigma_i(p) \mapsto (p, \lambda_1 \cdots \lambda_k)$.

(2) <u>One forms:</u>

We consider dual space of each fiber E_p of vector bundle E.

<u>e.g.</u> Dual bundle of TM is $T^*M := (TM)^*$ called cotangent space.

<u>Def</u>: A section ω of cotangent bundle is called one-form. Denote $\Omega^1(M) = \Gamma(T^*M)$ is set of all sections

<u>Rmk</u>: In local chart (U, φ). $\exists (\partial_i)$ basis of TM. $\Rightarrow$ we can get basic one form dx^i: $dx^i(\partial_j) = \delta_{ij}$. $\Rightarrow \forall \sigma \in \Omega^1(M)$. $\sigma = \Sigma \sigma^i dx_i$. where $\sigma^i = \sigma(\partial_i) \in C^\infty(U)$.

To construct one-form, we can also start from $f \in C^\infty(P)$.

which induces $D_p f : T_p M \to T_{f(p)} \mathbb{R}^1 \cong \mathbb{R}^1$.

Set $(d_p f)_p : T_p M \to \mathbb{R}^1$, define by

$d_p f(X_p) = X_p f$, $\forall p \in M$.

Rmk: It can also expressed in basis in local chart (U, φ). Def $x^i = z^i \circ \varphi$

Then: $\forall f \in C^\infty(M)$. $df = \sum \partial_i f \, dx^i$.

i) $dx^i(\partial_j) = \partial_j(z^i \circ \varphi) = \frac{\partial}{\partial x_j} z^i = \delta_{ij}$

2) $df(\partial_j) = \partial_j f$.

Cor. $df = 0 \iff f \equiv \text{const}$.

Pf: $\forall$ local chart (U, φ).

$\partial_j f = 0 . \forall j \iff \partial(f \circ \varphi^{-1}) / \partial x_i = 0$

e.g. consider $\theta : S^1 \to \mathbb{R}^1$ $\theta \in (\theta_0, \theta_0 + 2\pi)$

$(\cos\theta, \sin\theta) \mapsto \theta$

Set $d\theta$ is coordinate basis one-form. with local vector $(-\sin\theta, \cos\theta) \in TS^1$

$\Rightarrow \forall f^* \in T^*S^1$. $f^* = f d\theta$. for some f

Rmk: $\forall n$. TS^n is globally trivial

i.e. $TS^n \simeq S^n \times \mathbb{R}^n$. So T^*S^n

is trivial.

Push Back:

$f: M \to N$ induces $D_p f : T_p M \to T_{f(p)} N$

can then induce $(D_p f)^* : T^*_{f(p)} N \to T^*_p M$.

define by $(D_p f)^*(\omega_{f(p)}) = \omega_{f(p)}(D_p f(\cdot))$

e.g.: $i: M \hookrightarrow N$ is inclusion.

Then: $i^* \omega$ is just $\omega|_M$. restrict in one form.