# Bilinear Forms

(1) Bilinear form:

Def: i) $b: V \times V \to \mathbb{R}'$ is called bilinear form on $V$ if it's bilinear function.

it's symmetric or antisym if $b(w,v) = b(v,w)$ or $-b(v,w)$.

Rmk: i) Any $b$ can be decomposed into symmetric part $b^+$ plus anti-sym part $b^-$ uniquely.

by $b^{\pm}(w,v) = \dfrac{b(w,v) \pm b(v,w)}{2}$.

ii) linear map $L: W \to V$ can pull back $b$ on $V$ to $L^*b$ in $W$. by $L^*b(w_1, w_2) = b(Lw_1, Lw_2)$

ii) For a bilinear form $b$. We set the

quadratic form $q(\cdot)$ associated it by

denote : $q(v) \triangleq b(v,v)$.

Rmk: i) $q(\cdot)$ and $b(\cdot)$ are one-to-one

correspd. since $2b(v,w) = q(v+w)$

$- q(v) - q(w)$. ( So $q$ isn't linear )

ii) $q(\cdot)$ only depends on its sym

part ( $b^-(v,v) = -b^-(v,v) = 0$ )

So : $\dim V = k \Rightarrow \dim Q(V) = r = \frac{k(k+1)}{2}$. where $Q(V)$ is space of

quadratic forms on $V$.

( Write in matrix. $q(v) = v^T (b_{ij}) v$ )

iii) We call bilinear form $b$ is positive

semidefinite / definite if $q(v) \geq 0 / > 0$.

for $\forall v \neq 0$.

We call positive definite form on $V$

by inner product. on $V$.

Rmk : $L^* b$ is positive semi-definite if $b$ is

For the case of positive definite. it requires $L$ is injection additional

Next, we focus on sym bilinear form:

Consider vector bundle $\pi: E \to M$. Set $E = TM$. and consider vector bundle $\pi'$:

$Q(TM) = \bigcup_p Q(T_pM) \longrightarrow M$. $q \in Q(T_pM) \mapsto p$.

<u>Def</u>: A positive definite section $g \in \Gamma(Q(TM))$ is called Riemannian metric.

<u>Rmk</u>: i) By one-to-one correspd with bilinear form. We can identify it as inner product on each $T_pM$:

$\langle X_p, Y_p \rangle := g_p(X_p, Y_p)$. $X, Y \in \mathfrak{X}(M)$

Then in local chart $(U, \varphi)$. $g$ is given by $g_{ij} := g(\partial_i, \partial_j) \in C^\infty(U)$

$\Rightarrow (g_{ij})$ is sym. positive definite.

ii) By the argument of pull-back:

For $f: M \to N$. We def $:(f^* g)(X_p, Y_p)$

$= g(f_* X_p, f_* Y_p)$.

It still remain a Riemannian metric

$\Leftrightarrow$ $f$ is a immersion.

e.g. Std Riemannian metric $g \stackrel{\triangle}{=} (g_{ij}) = (\delta_{ij})$.

under std chart $(\mathbb{R}^m, id)$. on $\mathbb{R}^m$.