# Partitions of Unity

Note that there's always zero section $\sigma_p = 0 \in E_p$. And we wonder whether there is nowhere vanishing section, which is related to the existence of Riemannian metric.

e.g. Trivial bundle $M \times \mathbb{R}^r$ has many nowhere vanishing section. (nonzero const. section)

But other times, no nonvanishing section of $TM$ can happen (e.g. Hopf index thm. $S^1$)

Rmk: As for $\otimes(TM)$, for $\forall M$, nonvanishing section always exists.

So Riemannian metric always exists.

(1) Definitions:

Def. i) $\{S_\alpha\}$ subsets of t.s. $X$. is locally finite if $\forall p \in X$. $\exists U_p$. nbd, intersects finite $S_\alpha$.

ii) (Smooth) partition of unity on $M$ is collection of (smooth) $\psi_\alpha : M \to \mathbb{R}^1$. s.t.

(a) $\psi_\alpha \geq 0$. (b) $\{\psi_\alpha\}_A$ is locally finite.

(c) $\sum_A \psi_\alpha \equiv 1$

Rmk: i) e.g. $\psi \equiv 1$. (trivial case)

ii) $\forall p$. $\exists U_p$. $\sum_A \psi_\alpha$ is finite sum on $U_p$. $\Rightarrow$ $\sum_A \psi_\alpha$ is smooth.

iii) Open cover $\{U_\alpha\}_A$. PoU $\{\psi_\beta\}$ is said subordinate to $\{U_\alpha\}_A$. if $\forall \beta$. $\exists \alpha(\beta)$. s.t. $\operatorname{Supp} \psi_\beta \subset U_{\alpha(\beta)}$

Rmk: We can define new PoU by

Set $\bar{\psi}_\alpha = \sum_{\alpha(\beta)=\alpha} \psi_\beta$.

iv) Open cover $\{V_\beta\}$ is a refinement of another open cover $\{U_\alpha\}$. if $\forall \beta$. $\exists \alpha(\beta)$. s.t. $V_\beta \subset U_{\alpha(\beta)}$.

t.s. $X$ is paracpt if $\forall$ open cover for $X$ has a locally finite refinement.

Rmk: i) paracpt is weakened in sense: replacing "finite refinement" by "locally finite refinement".

ii) $\mathbb{R}'$ is paracpt. not cpt.

Any metric space is paracpt.

iii) $C_2$ + Hausdorff + locally cpt $\Rightarrow$ paracpt. + (Hausdorff) $\Rightarrow$ normal.

Some replaces $C_2$ by paracpt to define mfd. But it may have uncountable components. (e.g. $\mathbb{R}/\mathbb{R}' \times \mathbb{R}'$)

Prop. $\forall$ open cover $\{U_\alpha\}_\alpha$ of cpt mfd $M^n$. $\exists$ PoU subordinate to it.

Pf: $\forall p \in M$. $\exists \alpha$. s.t. $p \in U_\alpha$. find $f_p$ supports on $U_\alpha$. and $U_p \overset{\Delta}{=} \{f_p > 0\} \subset U_\alpha$.

$\Rightarrow \exists (U_{p_i})_1^n$ cover $M$. by cpt.

Set $f_k = f_{pk} / \sum_1^n f_{pi}$

Lemma[1] $\forall$ mfd $M$ has a countable basis consisting of coordinate nbds with cpt closure.

Pf: From countable basis $(B_k)$.

Set $\mathscr{B} = \{ B \in (B_k) \mid B$ is contained in some coordinate nbd, have cpt closure $\}$.

Prove: subcollection $\mathscr{B}$ is still a basis.

$\forall W \underset{open}{\subseteq} M$, $p \in M$. $\exists (U, \varphi)$. of $p$. st. $U \subset W$. $\varphi(p) = 0$. $B_2(0) \subset \varphi(U)$.

$\Rightarrow V = \varphi^{-1}(B_1(0))$ has cpt closure.

Note $\exists B_i \in (B_k)$. st. $p \in B_i \subset V$.

So $B_i$ also $\in \mathscr{B}$.

Lemma[2]. $\forall$ mfd $M$. has a cpt exhaustion $(W_k)$.

i.e. $\emptyset \neq W_1 \subset \overline{W_1} \subset W_2 \subset \overline{W_2} \subset \cdots$. st. $\forall k$.

$W_k$ is open. $\overline{W_k}$ is cpt. $\bigcup_i W_i = M$.

**Pf:** Let $\mathcal{B} = (B_j)$ is basis in $M'$.

Choose $(i_k)$. $i_0 = 1$. St. $W_k = \bigcup_i^{i_k} B_j$. &

$\overline{W_{k-1}} \subset W_k$. (It's possible by cpt.)

Also: $\bigcup B_j = M \subset \bigcup W_k = M$.

**Cor.** For mfd $M$. We can find countable family of subsets $K_i \subset O_i \subset M$. St. $K_i$ cpt, $O_i$ open and $\bigcup K_i = M$. $(O_i)$ is locally finite.

**Pf:** Let $(W_i)$ is cpt exhaustion above

Let $K_i = \overline{W_i} / W_{i-1}$. $O_i = W_{i+1} / \overline{W_{i-2}}$.

Note $O_i \cap O_j = \emptyset$. if $|i-j| > 2$.

**Cor.** Any mfd $M$ is paracpt.

**Pf:** Fix $(K_i)$ and $(O_i)$ above.

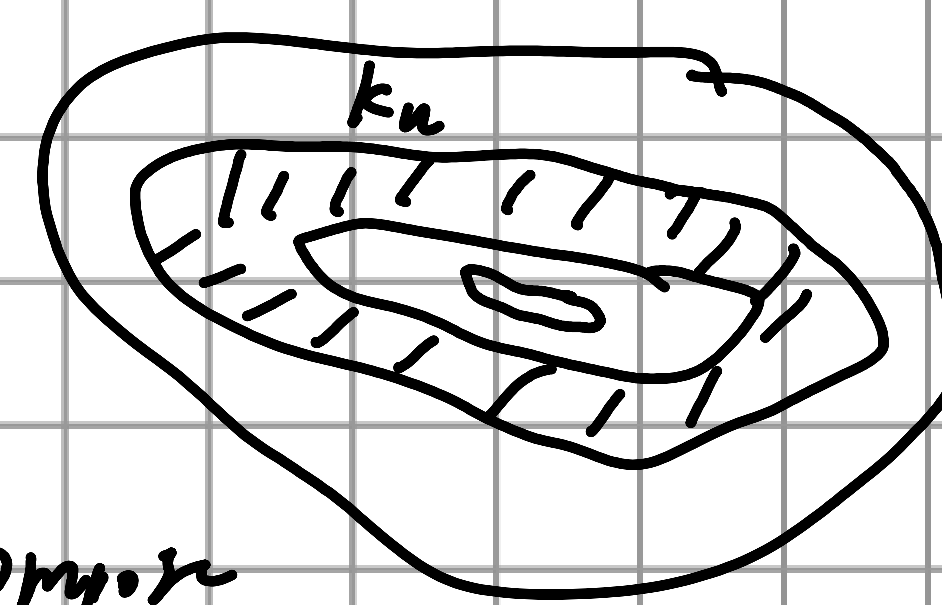
For $\forall (U_\alpha)$ open cover of $M$.

$\forall i$. $K_i$ can be covered by finite $U_\alpha \cap O_i$. rename it by $O_i^j$ for $j = 1, 2, \cdots i_k$. $\Rightarrow (O_i^j)_{i,j}$ is LFR.

Procedure: i) Find precpt coordinate Basis.

ii) Construct cpt exhaustion.

iii) Construct cpt / locally finite open bands




iv) For each mfd $M$. We can decompose it on each $K_n$ to discuss.

$(O_i)$ can decompose open set.

Thm. $\forall (U_\alpha)$ cover of mfd $M$. $\Rightarrow \exists$ PoU of $(\psi_i)$ subordinate to $(U_\alpha)$.

Pf: Fix $K_i$ and $O_i$ constructed above

$\forall p \in K_i$. $\exists \alpha(p)$. s.t. $p \in U_{\alpha(p)} \cap O_i$.

Note $\exists f_p \in C^\infty$. $f_p \geq 0$. supports on $U_{\alpha(p)} \cap O_i$. Set $V_p = \{f_p > 0\}$.

$\Rightarrow \exists$ finite $(V_{p_k^i}) < O_i$ cover $K_i$

So we get locally finite family $\{f_{k,i}\}_{k,i}$. Set $\psi_{k,i} = f_{k,i} / \sum_{k,i} f_{k,i} \neq 0$

Rmk: In the construction from $(K_i)$, $(O_i)$ above. We find $\forall K \underset{cpt}{\subseteq} M$. only intersects finite many $\operatorname{supp}\psi_i$ since $\forall p \in K$. $\exists U_p$ intersect finite $\operatorname{supp}\psi_i$. $K$ can be cover finitely.

(2) Application:

Thm. $\forall$ mfd $M^n$ admits a Rieman metric.

Rmk: Idea is for $\pi: E \to M$. Find $(\psi_\alpha)$ P.o.U subordinate to cover $\{U_\alpha\}$ for local section $\sigma_\alpha \in \Gamma(E_{U_\alpha})$. We can glue them up to get a global section by P.o.U: $\sum \psi_\alpha \sigma_\alpha$.

Pf: If $g_0$ is std Rieman metric. Set $\{(U_\alpha, \varphi_\alpha)\}$ is atlas of $M$. We get local Ric-metric by set $g_\alpha = \varphi_\alpha^*(g_0)$ on $U_\alpha$.

Then find p.u $\{\psi_\alpha\}$ subordinate to $\{U_\alpha\}$. $\Rightarrow$ We have: $\sum \psi_\alpha g_\alpha$ is the global Rieman metric on $M$.

<u>Thm</u>. Any cpt mfd $M^m$ can be embedded in $\mathbb{R}^n$. for some $n$.

<u>Pf</u>: Set $f_p : M \to \mathbb{R}^1$. s.t. $f_p \equiv 1$ on $V_p \subset U_p$. $(U_p, \varphi_p)$ is a local chart.

Then cover $M$ by finite $(V_{p_i})_1^k$.

Set $f(p) = (f_1(p)\varphi_{p_1}(p), \dots f_k(p)\varphi_{p_k}(p)$

$, f_1(p) \dots f_k(p))$.

$: M \to \mathbb{R}^{k(m+1)}$ is an embedding.

<u>Rmk</u>: $\forall$ mfd $M^m$ can be embedd into $\mathbb{R}^{2m}$:

We can use the $(K_i)$ decomp. and embed each part into $\mathbb{R}^n$.

(Witney trick $n$ can be $2m$)

Then inherit Rieman metric of $\mathbb{R}^n$

Which can also prove RM exist.