Riemannian Mfd.

Def: Rieman mfd (M, g) is mfd M^n equiped with fix Rieman metric g.

Next, we denote $\langle X_p, Y_p \rangle = g_p(X_p, Y_p)$ and $\|X_p\| = \langle X_p, X_p \rangle^{\frac{1}{2}}$.

Def: i) $\gamma : [a,b] \to M$ is piece-smooth. length of γ is $\text{len}(\gamma) \stackrel{\Delta}{=} \int_a^b \|\gamma'(t)\| dt$.

Rmk: $\text{Len}(\gamma)$ is invar. under reparametized

Set $s(t) = \int_a^t \|\gamma'(s)\| ds$. and

$\tilde{\gamma}(t) = \gamma_{s^{-1}(t)} \Rightarrow \|\tilde{\gamma}'(t)\| = 1.\ \forall t > 0$

for γ is piecewise immersion.

ii) distance $d(p,q) := \inf \{ \text{len}(\gamma) \mid \gamma$ is piecewise smooth from p to q in $M \}$.

Rmk: i) $d(p,q) \stackrel{\Delta}{=} +\infty$ if p, q are in different mfd components.

ii) $d(p,r) \leq d(p,q) + d(q,r)$. by concatenating smooth paths

And $d(p,q) = d(q,p)$. also holds.

iii) Note if $f: N \to M$ immersion

$\Rightarrow f^*g$ is also Rie-metric on N.

$\mathrm{len}_{f^*g}(\gamma) = \mathrm{len}_g(f\circ\gamma)$ by def.

So: $d_{f^*g}(p,q) \geq d_g(p,q)$.

("=" holds if f is diffeo. f_* is iso.)

<u>Lemma</u>. Equip $\mathbb{R}^n$ with standard Riemannian metric g_0. Then: $d(p,q) = \|p-q\|$.

<u>Rmk</u>: With Rmk ii) above. we see d is a true metric in $\mathbb{R}^n$.

<u>Pf</u>: By trans. invariant in $\mathbb{R}^n$. WLOG. set $q=0$. Note the straight line from p to 0 has length $\|p\|$.

Next. we want to show: $\forall \gamma$. piece-wise smooth. $\gamma(0)=p$. $\gamma(1)=0$. has at least length $\|p\|$.

WLOG. Set $\gamma(t) \neq 0$ on $t<1$. Otherwise:

Set $\tilde{\gamma} = \gamma|_{[0,t]}$ which has shorter len.

Let $\alpha(t) = \|\gamma(t)\|$. $\beta(t) = \gamma(t)/\|\gamma(t)\|$

So: $\gamma(t) = \alpha(t)\beta(t)$. $\Rightarrow \gamma' = \alpha'\beta + \alpha\beta'$.

$\|\gamma'\|^2 \geq |\alpha'|^2$. ($\langle\beta,\beta\rangle = 1 \Rightarrow \langle\beta',\beta\rangle = 0$)

$\Rightarrow \operatorname{len}(\gamma) \geq \left|\int_0^1 \alpha'(t)dt\right| = \|p\|$.

<u>Lemma.</u> If g is a Riemann metric on $U \underset{\text{open}}{\subseteq} \mathbb{R}^m$.

$K \subset U$. cpt. Then $\exists$ const. $0 < c \leq C$. st.

$c\|v\| \leq g(v,v)^{\frac{1}{2}} \leq C\|v\|$. $\forall v \in T_pU$. $p \in K$.

<u>Rmk</u>: Specially. $\forall \gamma \subset K$. from p to q. We

have $c\|p-q\| \leq c\operatorname{len}_{g_0}\gamma \leq \operatorname{len}_g\gamma \leq C\operatorname{len}_{g_0}\gamma$.

<u>Pf</u>: Note $\forall p \in K$. $\exists c_p. C_p$. $c_p\|v\| \leq g_p(v,v)^{\frac{1}{2}} \leq C_p\|v\|$.

Actually $c_p. C_p$ conti. at p.

We can set $c = \sup_K c_p$. $C = \sup_K C_p$.

<u>Lemma.</u> For Riemann mfd (M, g). its distance

d associated with g is truly metric.

Pf: For $p \neq q \in M$. $\exists (U_p, \varphi)$ local chart.

s.t. $\varphi(p) = 0$. $\varphi(U_p) \supset \overline{B_1(0)}$. $\varphi(q) \notin B_1(0)$

So: $\forall$ path γ connects p, q piece-smooth

$$\mathrm{len}_g \gamma = \mathrm{len}_{(\varphi^{-1})^* g} \varphi \circ \gamma \overset{\text{Lem}}{\geq} C \cdot 1$$

i.e. $d_g(p, q) \geq C$ if $q \notin \varphi^{-1}(B_1(0)) \subset U_p$.

Cor. The metric topo. of (M, d) agrees with the original given topo. of M.

Pf: 1) By proof above $\forall p \in U$. $\exists U_p$ local chart of p s.t. $U_p \subset U$.

Then: $d_g(p, q) < C \Rightarrow q \in U_p$.

So $\{q \mid d_g(p, q) < C\} \subset U_p$.

2) $\forall p \in W = B^g(R)$. Let $\varepsilon > 0$. $p \in B^g(C\varepsilon) \subset W$. For local chart (U_p, φ):

Set $U_\varepsilon = \varphi^{-1}(B^{g_0}_{\varphi(p)}(\varepsilon)) \cap U_p$.

$\forall q \in U_\varepsilon$. Fix γ is straight line conn. $\varphi(p), \varphi(q)$. $d_g(p, q) \leq \mathrm{len}_g \varphi^{-1}(\gamma) = \mathrm{len}_{(\varphi^{-1})^* g} \gamma \leq C\, \mathrm{len}_{g_0} \gamma < C\varepsilon \Rightarrow q \in W$.

$$\mathrm{len}_{g_0} \gamma = \|\varphi(p) - \varphi(q)\|$$

Rmk: We just use $\mathrm{len}_g \sim \mathrm{len}_{g_0}$ in $\forall K \underset{\text{cpt}}{\subset} \mathbb{R}^n$.