Intro. & Examples

<u>Denote</u>: Consider on $\mathbb{R}^d$. $V, W \in \mathbb{R}^d$. $Z \in \mathbb{R}^{d\times d}$

$\mathrm{div}(V) = \sum_{i}^{d} \partial_j V_j$. $\nabla V = (\partial_i V_j)_{i,j}$.

$V \otimes W = (V_i W_j)_{i,j}$. $DV = \frac{1}{2}(\nabla V + \nabla V^T)$.

$\mathrm{div}(Z) = (\sum_j \partial_j Z_{ij})_{i=1\ldots d}$

<u>Rmk</u>: For $x \in \mathbb{R}^1$. replace V_j by x above.

① <u>Stokes equation</u>:

It's to model the laminar flow of a Newtonian field through bdd domain Ω under influence of a force.

$-\Delta \vec{V} + \nabla \pi = f$. in Ω (0.1)

$\mathrm{div}\,\vec{V} = 0$ in Ω (0.2)

$\vec{V} = 0$ on $\partial\Omega$ (0.3)

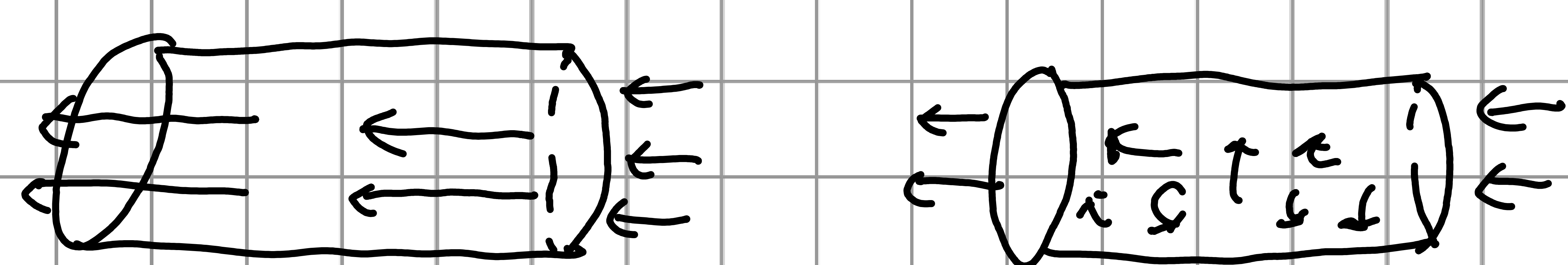
where $f: \Omega \to \mathbb{R}^d$ is given force. $\vec{V}$ is the velocity vector field : $\Omega \to \mathbb{R}^d$. π is the kinematic pressure : $\Omega \to \mathbb{R}^1$.

(0.3) is no-slip boundary condition.

(The fluid won't move out of boundary).

(0.2) : The fluid is incompressible.

(0.1) describes shear-stress acting on the fluid influenced by the force.



Laminar (Non-turbulent)

Turbulent

Rmk: It can be solved by Lax-Milgram

② p-Laplace equation:

It's to model the deflection of non-linear elastic membrane fixed at the bdy of a domain under influence of a force.

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega. \quad (1.1)$$

$$u = 0 \quad \text{on } \partial\Omega. \quad (1.2)$$

deflection field $u : \Omega \to \mathbb{R}^1$.

(1.1) describes deflection of membrane influenced by force $f: \Omega \to \mathbb{R}^1$.

(1.2) says the membrane is fix on bdy

Rmk: Poisson equation is to model the linear elastic membrane, which's special case when $p = 2$.

③ P - Stokes:

We consider ① is on Non-Newtonian fluid, then the equations become:

$$-\operatorname{div}(S(D\vec{v})) + \nabla \pi = f. \quad \text{in } \Omega.$$

$$\operatorname{div}(\vec{v}) = 0. \quad \text{in } \Omega.$$

$$\vec{v} = 0. \quad \text{on } \partial\Omega.$$

Where $S(D\vec{v}): \Omega \to \mathbb{R}^{n\times n}_{sym}$ is extra-stress tensor depends on strain-rate tensor $D\vec{v} = \frac{1}{2}(\nabla\vec{v} + \nabla\vec{v}^T)$. $f: \Omega \to \mathbb{R}^n$. force.

Rmk: ② and ③ are non-linear. So the Lax-Milgram Lemma won't work.

④ <u>p-Navier-Stokes:</u>

If we want to model both laminar and turbulent flow in ③, we have:

$$-\operatorname{div}(S(D\vec{v})) + \operatorname{div}(\vec{v}\otimes\vec{v}) + \nabla \pi = f \quad \text{in } \Omega$$

$$\operatorname{div}(\vec{v}) = 0 \quad \text{in } \Omega$$

$$\vec{v} = 0 \quad \text{on } \partial\Omega$$

i.e. we add $\operatorname{div}(\vec{v}\otimes\vec{v}) = \left(\sum_{j=1}^{n} \partial_j (v_i v_j)\right)_i$ $: \Omega \to \mathbb{R}^d$. the convective term to model turbulent in fluid. where $\vec{v} \otimes \vec{v} = (v_i v_j)_{ij}$.

① <u>Subseq. Conv. Principle:</u>

Recall X is reflexive $\iff$ $\forall$ bdd seq admit a weakly convergence subseq.

<u>Lemma.</u> X is reflexive Banach. Then: if a bdd seq x_n has same limit for its any weakly conv. subseq. $\Rightarrow$ $x_n \rightharpoonup x$ (proved by contradiction)