Monotone Operator Theory

The objective is to generalize the following IFT in $X = \mathbb{R}^1$ to infinite dimensional Banach space

<u>Thm</u>: $A : \mathbb{R}^1 \to \mathbb{R}^1$. Satisfies

i) $x_n \to x$ in $\mathbb{R}^1 \Rightarrow Ax_n \to Ax$. (Conti.)

ii) $Ax \to \pm\infty$ if $x \to \pm\infty$ (Coercive)

$\Rightarrow$ i) A is surjective

ii) If additionally, $x > y \Rightarrow Ax > Ay$. Then: A is bijective and A^{-1} is conti.

<u>Pf</u>: i) By intermediate value Thm. and condition i)

ii) injective is trivial. And:

For $y_n \to y$. $\Rightarrow A^{-1}y_n$ is bdd

otherwise, contradict with cond. ii).

$\Rightarrow A^{-1}y_n$ has convergent subseq.

Note $AA^{-1}y_{n_k} \to y$. (Note A is bijective)

Σ every subseq admits the same

limit if it converges.

$\Rightarrow A^{-1}y_n \to A^{-1}y$. A^{-1} is conti.

Next, we will generalize the conti.

condition i) and coercive condition ii).

(1) Conti. Condition:

Definition 2.2 (Notions of continuity and boundedness)
Let $(X, \|\cdot\|_X)$ be a (real) Banach space. Then, an operator $A\colon X \to X^*$ is called

(i) continuous *if for a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and an element $x \in X$ from*

$$x_n \to x \quad \text{in } X \quad (n \to \infty),$$

it follows that

$$Ax_n \to Ax \quad \text{in } X^* \quad (n \to \infty).$$

(ii) weakly continuous *if for a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and an element $x \in X$ from*

$$x_n \rightharpoonup x \quad \text{in } X \quad (n \to \infty),$$

it follows that

$$Ax_n \rightharpoonup Ax \quad \text{in } X^* \quad (n \to \infty).$$

(iii) demi-continuous *if for a sequence* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *and an element* $x \in X$ *from*

$$x_n \to x \qquad \text{in } X \quad (n \to \infty)\,,$$

it follows that

$$Ax_n \rightharpoonup Ax \quad \text{in } X^* \quad (n \to \infty)\,.$$

(iv) strongly continuous *if for a sequence* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *and an element* $x \in X$ *from*

$$x_n \rightharpoonup u \qquad \text{in } X \quad (n \to \infty)\,,$$

it follows that

$$Ax_n \to Au \quad \text{in } X^* \quad (n \to \infty)\,.$$

(v) hemi-continuous *if for every* $x, y, z \in X$*, the function*

$$(t \mapsto \langle A(x+ty), z\rangle_X)\colon [0,1] \to \mathbb{R}$$

is continuous.

(vi) radially continuous *if for every* $x, y \in X$*, the function*

$$(t \mapsto \langle A(x+ty), y\rangle_X)\colon [0,1] \to \mathbb{R}$$

is continuous.

(vii) locally bounded *if for every* $x \in X$*, there exist constants* $\varepsilon = \varepsilon(x), M = M(x) > 0$ *such that*

$$\|Ay\|_{X^*} \le M \quad \text{for all } y \in \overline{B}_\varepsilon^X(x)\,,$$

where $\overline{B}_\varepsilon^X(x) := \{y \in X \mid \|y - x\|_X \le \varepsilon\}$.

ix) A is bdd if $\exists c>0$. s.t. $\forall x$. $\|Ax\|_Y \le c\|x\|_X$.

$\Leftrightarrow$ A is bdd $\Leftrightarrow$ A conti.
A is LO   A is LO

$\not\Rightarrow$ A is bdd $\not\Rightarrow$ A conti.

Rmk: LO $A: X \to Y$ is conti $\Leftrightarrow$ weakly conti.

($\Rightarrow$) $\exists A^* \in \mathcal{L}(Y^*, X^*)$. ($\Leftarrow$) By contradict! (*)

(Actually A^* exists $\Leftrightarrow$ A is BLO, for ($\Rightarrow$). We can apply closed graph Thm)

(*) We use A is bdd $\Leftrightarrow$ A is conti.

Lemma 2.4 (relations between notions of continuity and boundedness)
Let $(X, \|\cdot\|_X)$ *be a (real) Banach space and* $A\colon X \to X^*$ *an operator. Then, the following statements apply:*

(i) The following implications apply:

X reflex $\Rightarrow$ cpt

A linear

$$\begin{tikzcd}
 & \textit{strong continuous} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\textit{weakly continuous} \arrow[rr, leftrightarrow, "A \text{ linear}"] \arrow[dr, Rightarrow] & & \textit{continuous} \arrow[dl, Rightarrow] \\
 & \textit{demi-continuous} \arrow[d, Rightarrow] & \\
 & \textit{hemi-continuous} \arrow[d, Rightarrow] & \\
 & \textit{radially continuous} &
\end{tikzcd}$$

$\Rightarrow$ local bdd

ii) A is strongly conti. X reflexive $\Rightarrow$ A is cpt.

iii) A is demi-conti. $\Rightarrow$ A is locally bdd

iv) A is linear, cpt $\Rightarrow$ A is strongly conti.

Pf: ii) $\forall M \subseteq X$. bdd set. $\forall (y_n) \subseteq A(M)$.

Set $Ax_n = y_n$. With reflexive:

$\exists (x_{n_k}) \subseteq (x_n) \rightharpoonup x$.

So: $(Ax_{n_k}) = (y_{n_k}) \to Ax$.

iii) By contradiction. $\exists (x_n) \to x \in X$

so. $\|Ax_n\| \to +\infty$.

But demi-conti $\Rightarrow$ $Ax_n \rightharpoonup Ax$.

$\Rightarrow$ (Ax_n) is bdd. contradict!

iv) For $(x_n) \rightharpoonup x$. $\overset{A \text{ linear}}{\underset{\text{cpt}}{\Longrightarrow}}$ $Ax_n \rightharpoonup Ax$.

With cptness. $\exists (x_{n_k})$. $Ax_{n_k} \to x^*$.

$\Rightarrow$ $x^* = Ax$. And any subseq has same limit by this argument

So: $Ax_n \to Ax = x^*$.

(2) Monotonicity Cond:

Note in $X = \mathbb{R}^1$. the monotonicity also depends on $(\mathbb{R}^1, <)$ is totally order.

But we can also see:

$A: \mathbb{R}^1 \to \mathbb{R}^1$. strictly mono. $\Leftrightarrow \forall x \neq y \in \mathbb{R}^1$. we have: $(Ax - Ay)(x-y) > 0$

Definition 2.5 (Notions of monotonicity)
Let $(X, \|\cdot\|_X)$ be a (real) Banach space. Then, an operator $A\colon X \to X^*$ is called

(i) **monotone** *if for every $x, y \in X$, it holds that*

$$\langle Ax - Ay, x - y\rangle_X \geq 0.$$

(ii) **strictly monotone** *if for every $x, y \in X$ with $x \neq y$, it holds that*

$$\langle Ax - Ay, x - y\rangle_X > 0.$$

(iii) **d-monotone** *if there exists a strictly non-decreasing function $\alpha\colon [0, +\infty) \to \mathbb{R}$ such that for every $x, y \in X$, it holds that*

$$\langle Ax - Ay, x - y\rangle_X \geq \big(\alpha(\|x\|_X) - \alpha(\|y\|_X)\big)(\|x\|_X - \|y\|_X).$$

(iv) **uniformly monotone** *if there exists a strictly non-decreasing function $\rho\colon [0, +\infty) \to \mathbb{R}$ with $\rho(0) = 0$ such that fir every $x, y \in X$, it holds that*

$$\langle Ax - Ay, x - y\rangle_X \geq \rho(\|x - y\|_X). \geq \rho(\|x\| - \|y\|)$$

(v) **strongly monotone** *is there exists a constant $m > 0$ such that for every $x, y \in X$, it holds that*

$$\langle Ax - Ay, x - y\rangle_X \geq m\,\|x - y\|_X^2.$$

Rmk: For $p \in (1,2)$, $m > 0$. There's no operator $A: X \to X^*$. satisfies $\langle Ax - Ay, x-y\rangle \geq m\|x-y\|^p$. By contradiction: $\forall N \geq 1$

$$m\|x-y\|_X^p \leq N^{p-1}\sum_{i}^{N} m\|(x+\tfrac{i}{N}(y-x)) - (x+\tfrac{i+1}{N}(y-x))\|^p$$
$$\leq N^{p-2}\langle Ax - Ay, x-y\rangle_X.$$

Lemma 2.6 (relations between notions of monotonicity)

Let $(X, \|\cdot\|_X)$ be a (real) Banach space. Then, the following implications apply:

$$\begin{tikzcd}[column sep=small]
 & \text{strongly monotone} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] \arrow[r, Rightarrow] & \text{Coercive.} & \\
\text{d-monotone} \arrow[dr, Rightarrow, "X \text{ stricly convex}"] & & \text{uniformly monotone} \arrow[dl, Rightarrow] \arrow[r, "+\,X \text{ reflex}"] & \text{demi-conti.} \\
 & \text{strictly monotone} \arrow[d, Rightarrow] & & \\
\text{locally bdd} & \text{monotone} \arrow[l] \arrow[r, "+\text{radial conti.}", "\text{(Minty)}"'] & \text{Type (M)} \\
 & & \text{max-mono.}
\end{tikzcd}$$

<u>Rmk</u>: i) X is strictly convex if for $x \neq y \in X$. $\|x\| = \|y\| = 1 \Rightarrow \|\frac{x+y}{2}\| < 1$.

ii) Uniformly convex $\Rightarrow$ Strictly convex.

<u>Lemma.</u> (Minty's trick)

X is real Banach space. $A: X \to X^*$ is monotone and radial conti. Then:

i) A is maximal monotone. i.e. $(x, x^*) \in X \times X^*$. s.t. $\langle x^* - Ay, x - y\rangle \geq 0$. $\forall y$ $\Rightarrow x^* = Ax \in X^*$.

ii) A is Type (M). i.e. $\forall (x_n) \subset X$. $x^* \in X^*$. $x_n \rightharpoonup x$. $Ax_n \rightharpoonup x^*$. $\overline{\lim_{n\to\infty}} \langle Ax_n, x_n\rangle \leq \langle x^*, x\rangle$ $\Rightarrow Ax = x^*$.

<u>Pf</u>: i) Note set $y = x \pm ty$. divide t

We have $\Rightarrow \langle x^* - A(x+t\eta), \eta \rangle \geq 0$

Set $t \to 0$ follow from radial. conti.

iii) For $\forall \eta \in X$. Consider:

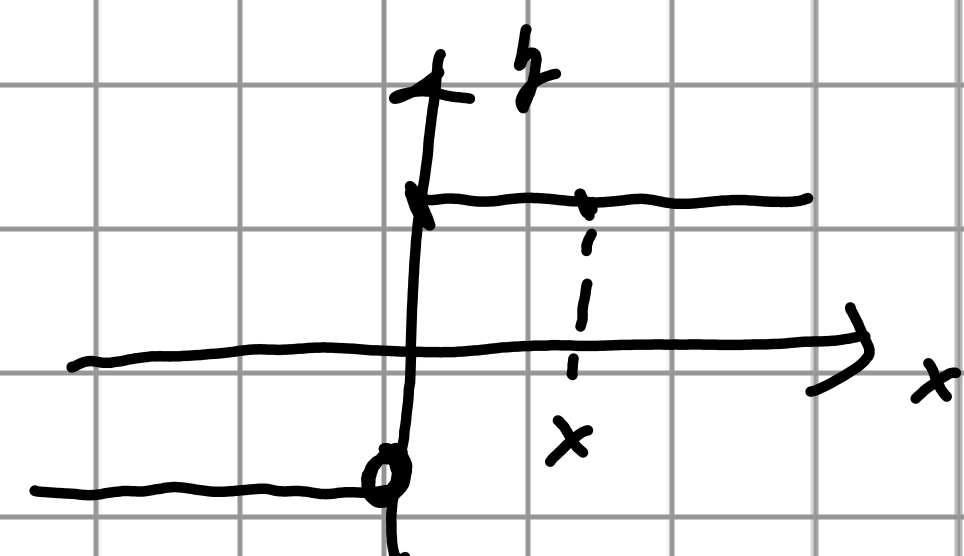
$\langle Ax_n - A\eta, x_n - \eta \rangle \geq 0$ (by mono.)

Take $\overline{\lim}$ on $\langle Ax_n, x_n \rangle \geq \langle Ax_n, \eta \rangle + \cdots$

$\Rightarrow \langle x^* - A\eta, x - \eta \rangle \geq 0$. So $x^* = Ax$.

Rmk: i) Maximal mono. means: there's no other monotone extension for $A|_{\mathcal{D}(A)}$.

e.g.



ii) A is also type $(M)^*$. i.e. replace " $\square$ " by $Ax_n \xrightarrow{*} x^*$. proved in same way

Lemma. $(X, \|\cdot\|_X)$ is real Banach. If $A: X \to X^*$ is mono. Then:

i) A is locally bdd.

ii) if A is linear additionally $\Rightarrow A$ is BLF

iii) if X is reflexive, A is radially conti.

additionally, then A is hemicont.

Pf: i) By contradiction. if $\exists (x_n) \to x \in X$.

$\|Ax_n\| \to \infty$.

Note: $\langle Ax_n, y-x \rangle \overset{\text{mono}}{\leq}$

$\langle Ax_n, x_n - x \rangle - \langle Ay, x_n - y \rangle$

$\leq \|Ax_n\| \|x_n - x\| + \|Ay\| (\|x_n\| + \|y\|)$

Set $a_n = (1 + \|Ax_n\| \|x_n - x\|)^{-1}$.

$A_n \triangleq a_n A x_n$

$\Rightarrow \langle A_n, y - x \rangle \leq 1 + c(y, x)$.

Set $y = x + y$. $S_n := \sup_n \| \langle A_n, y \rangle | < \infty$

By UBP. $\Rightarrow \|A_n\| \leq c(x)$.

But $\exists m, \forall n \geq m$. s.t. $\|x - x_n\| \leq \frac{1}{2c(x)}$.

$\Rightarrow \|Ax_n\| \leq c(x)/a_n = c(x)(1 + \square)$

$\leq c(x) + \|Ax_n\|/2$.

So: $\sup_{n \geq m} \|Ax_n\| \leq 2c(x)$. Contradict!

ii) follows from i): $\|A \sum \frac{y}{\|y\|}\| \leq M < \infty$.

iii) For $x_n \to x$ in X, since A is locally bdd by i). So: $\exists (x_{n_k})$ s.t. $Ax_{n_k} \rightharpoonup x^*$ by reflexive.

$\Rightarrow \langle Ax_{n_k}, x_{n_k} \rangle \longrightarrow \langle x^*, x \rangle$.

So $\overline{\lim} \langle Ax_{n_k}, x_{n_k} \rangle = \langle x^*, x \rangle$.

Apply Minty's Lemma $\Rightarrow Ax = x^*$.

It holds for any convergent subseq.

$\Rightarrow$ So $Ax_n \rightharpoonup Ax$.

(3) <u>Coercivity Cond.</u>:

It's still not trivial to generalize the coercivity cond. Since it relys on the fact $(\mathbb{R}, <)$ is totally order in $\mathbb{R}^1$.

<u>Lemma</u> $A: \mathbb{R}^1 \to \mathbb{R}^1$ is coercive $\Leftrightarrow$

$(Ax) \cdot x / |x| \to +\infty$ if $|x| \to +\infty$.

<u>Pf</u>: $(\Leftarrow)$ is easy to check

($\Rightarrow$) For $\forall |x_n| \to +\infty$. i) $x_n \to +\infty$ 2) $x_n \to -\infty$. 3°) $\exists (n_{k_1}) (n_{k_2})$. s.t. $x_{n_{k_1}} \to +\infty$. & $x_{n_{k_2}} \to -\infty$. In all three case they all hold:

since $F = (Ax)\,\mathrm{sgn}(x)$.

<u>Def</u>: $(X, \|\cdot\|_X)$ is Banach space. $A: X \to X^*$ is called coercive if $\frac{\langle Ax, x\rangle}{\|x\|} \xrightarrow{\|x\| \to \infty} \infty$

<u>Rmk</u>: It's equi.: $\exists \gamma: \mathbb{R}_{\geq} \to \mathbb{R}^1$. s.t. $\langle Ax, x\rangle \geq \gamma(\|x\|)\|x\|$ and $\gamma(s) \xrightarrow{s\to\infty} \infty$.

<u>Lemma</u> $A: X \to X^*$ is strongly mono. $\Rightarrow$ A is coercive.

<u>Pf</u>: Note $\langle Ax - Ay, x - y\rangle \geq m\|x - y\|^2$

Set $y = 0$. So:

$\langle Ax, x\rangle / \|x\| \geq m\|x\| - \|A0\|$

(*) <u>Main Theorem</u>:

<u>Thm.</u> (Browder - Minty)

$(X, \|\cdot\|)$ is separable. reflexive Banach.

If $A: X \to X^*$ is mono. radially conti.

and coercive. Then:

i) A is surjective. And $\forall x^* \in X^*$, we have $\mathbb{L}(x^*) := A^{-1}\{x^*\}$ is convex. closed. bounded.

ii) If additionally, A is strictly mono. Then A is bijection. $j_X \circ A^{-1}: X^* \to X^{**}$ is strictly mono. bdd. demi-conti.

iii) If additionally, A is strongly mono. Then: $j_X \circ A^{-1}: X^* \to X^{**}$ is Lipschitz.

iv) If additionally, A is strongly mono. and Lipschitz, then: $j_X \circ A^{-1}$ is strongly mono

Rmk: i) We can drop out the separable cond. here. It's for using Galerkin's finite dimension approxi.

ii) The proof is nonconstructive since we use fix pt Thm and subseq.

Pf: We first prove ii) and the result i):

1°) A is bijective:

Note A is strictly mono. So:

A is injective. With conseq. of i)

2) Strictly mono.:

By bijection: $\forall x^*, y^* \in A^*$. $\exists x, y \in A$. s.t.

$Ax = x^*$. $Ay = y^*$.

$\langle (j_X \circ A^{-1}) x^* - (j_X \circ A^{-1}) y^*, x^* - y^* \rangle$

$= \langle x^* - y^*, A^{-1}x^* - A^{-1}y^* \rangle = \langle Ax - Ay, x - y \rangle > 0$

3) Bdd:

If (x_n^*) is bdd. $\| (j_X \circ A^{-1}) x_n^* \| \to +\infty$

By coercive of A:

$\langle A(A^{-1}x_n^*), A^{-1}x_n^* \rangle / \|A^{-1}x_n^*\| \longrightarrow +\infty \quad (n \to \infty)$

But LHS $\leq \sup \|x_n^*\|$. which is contradict!

4) Demi-conti.

For $(x_n^*) \to x^*$ in X. By bdd of $j_X \circ A^{-1}$

$\Rightarrow (A^{-1}x_n^*)$ is bdd.

So $\exists x_{n_k}^*$. $A^{-1}x_{n_k}^* \rightarrow x$. by reflexive.

Apply Minty's Lemma, as before.

We have $Ax = x^* \Rightarrow x = A^{-1}x^*$.

It holds for $\forall$ convergent subseq.

$\Rightarrow A^{-1}x_n^* \rightarrow A^{-1}x$.

iii) & iv) By 2) in ii). We have:

$<(j_x \circ A^{-1})x^* - (j_x \circ A^{-1})y^*, x^* - y^*> =$

$<Ax - Ay, x - y>$

To prove i):

<u>Lemma</u>. (Browder Minty Thm for $\mathbb{R}^d$)

$f: \mathbb{R}^d \rightarrow \mathbb{R}^d$. conti. Then:

i) $\exists r > 0$. st. $\forall y \in \partial B_r^d(0)$. $f(y) \cdot y \geq 0$. $\Rightarrow$

$\exists x \in \overline{B_r^d(0)}$. st. $f(x) = 0$.

ii) f is coercive $\Rightarrow$ f is surjective.

iii) f is coercive, strictly mono. $\Rightarrow$ f is

bijection. f^{-1} is bdd. conti. strictly mono.

<u>Rmk</u>: i) It's generalization of $\mathbb{R}^1$ case we

proved at beginning.

ii) $f(r)\cdot r \geq 0$. on $\partial B_r^d(0)$ is called

generalized change of sign.

Note if $d=1$. it's equal: $\operatorname{sgn}(f(r))$

$\neq \operatorname{sgn} f(-r)$. $\Rightarrow f$ has root on $[-r,r]$.

<u>Lemma</u> (Brouwer fixed pt.)

$g: \bar{B}_r^d(0) \to \bar{B}_r^d(0)$ conti. $\Rightarrow g$

admits a fix pt. ($\exists x \in \bar{B}_r^d(0)$, $g(x)=x$)

<u>Pf</u>: ii). iii) is same as $\mathbb{R}^1$-case. For i):

By contradiction. if $|f(y)| \neq 0$. on $\bar{B}_r^d(0)$

Set $g(y) = -rf(y)/|f(y)|$

By Brouwer's fixed pt Thm:

$\exists\, X = g(x) = -r\, f(x)/|f(x)| \Rightarrow |x| = r$.

But $0 \leq X\cdot f(x) = -\frac{|f(x)|}{r}(g(x)\cdot x) = -r|f| < 0$

<u>Rmk</u>: In fact. i) $\Leftrightarrow$ Lem (Brouwer)

For converse: set $f(x) = x - g(x)$

<u>Pf of i)</u>:

Surjective:

Fix $x^* \in X^*$. We prove: $\exists x \in X$. s.t. $\forall y \in X$. we have $\langle Ax, y\rangle = \langle x^*, y\rangle$. (So: $Ax = x^*$).

Consider Galerkin system: (x_n) is o.n.b of X

Set $X_n = \text{span}\{x_1 \dots x_n\}$. So: $X = \bigoplus X_n$.

Next, we find sol. for $\forall n$. $X_n^* = (i_{X_n})^* X^*$.

$\exists z_n \in X_n$. s.t. $\langle Az_n, y_n\rangle = \langle x_n^*, y_n\rangle$. $\forall y_n \in X_n$.

Where $i_{X_n}: X_n \to X$. $(i_{X_n})^*: X^* \to X_n^*$. (Or say $A_n z_n = x_n^*$ in X_n)

i) Well-posedness:

Consider $\psi_n: \mathbb{R}^n \to X_n$, $(\beta_i)_1^n \mapsto \sum_1^n \beta_i x_i$.

Set $f_n: \mathbb{R}^n \to \mathbb{R}^n$. $\beta_n \mapsto (\langle A\psi_n(\beta_n) - x_n^*, x_i\rangle)_1^n$

f_n is conti. since A is hemi-conti.

f_n satisfies change of sign from coercive:

$$f_n(\beta_n)\beta_n = \langle A\psi_n(\beta_n) - x_n^*, \psi_n(\beta_n)\rangle$$
$$\geq (\gamma(\|\psi_n(\beta_n)\|) - \|x_n^*\|)\|\psi_n(\beta_n)\|.$$

$\gamma(x) \to +\infty$, if $x \to +\infty$. We can find a large ball $B^n(0,R)$. s.t. $f_n(\beta_n)\cdot\beta_n > 0$ on $\partial B^n(0,R)$.

So, apply Brouwer-Minty's Thm. $\exists \alpha_n \in \mathbb{R}^n$

s.t. $f_n(\alpha_n) = 0$. So: $\exists$ sol. $z_n = \varphi_n(\alpha_n) \in X_n$.

2) <u>Stability</u>:

We first investigate bddness of solution:

If $\bigcup_n (z_n^k)_k$ solutions of $A z_n^k = x_n^*$ on X_n. $\forall n$.

$\exists \; \|z_{n_k}\| \to \infty$. But $\dfrac{\langle A z_n, z_n \rangle}{\|z_n\|} = \dfrac{\langle x_n^*, z_n \rangle}{\|z_n\|} \le \|x^*\|$

contradict with coercive.

Next, we see (z_n) is bdd under A-image

since A is locally bdd. i.e. $\exists K, \varepsilon > 0$. s.t.

$\|Ay\| \le K$. for $y \in \bar{B}^X_\varepsilon(0)$.

By mono. of A: for $\forall y \in \bar{B}^X_\varepsilon(0)$. we have:

$\langle Az_n, y \rangle \le \langle A z_n, z_n \rangle - \langle Ay, z_n - y \rangle$

$\le (\|x^*\| + K)(\sup_n \|z_n\| + \varepsilon) \Rightarrow \|A z_n\| \le D/\varepsilon$.

3) <u>Weak convergence</u>:

Next, We want to prove $z_n \rightharpoonup z$. z is

solution of $Ax = x^*$. where $\langle Az_n, y_n \rangle = \langle x_n^*, y_n \rangle$.

$\forall y_n \in X_n$, $X_n^* = (i_{X_n})^* x^* \in X_n^*$

By reflexive: $\exists (z_{n_k}) \rightarrow z$ & $Az_{n_k} \rightarrow g^*$

$\langle g^*, y \rangle = \lim_{k\to\infty} \langle Az_{n_k}, y_{n_k} \rangle$. $(\exists y_n \in X_n \rightarrow y \in X)$

$= \lim_{k\to\infty} \langle X_{n_k}^*, y_{n_k} \rangle = \lim_{k\to\infty} \langle X^*, i_{X_{n_k}} y_{n_k} \rangle = \langle X^*, y \rangle$

So: $g^* = x^*$. Let $y = z$ above. We have:

$\overline{\lim} \langle Az_{n_k}, z_{n_k} \rangle = \langle X^*, z \rangle$. By Minty's trick.

$\Rightarrow Az = X^*$. i.e. A is surjective.

Convexity:

For $Ax = Ay = x^*$. By monotonicity of A:

$\langle X^* - Az, (\lambda X + (1-\lambda)y) - z \rangle$

$= \lambda \langle AX - Az, X - z \rangle + (1-\lambda) \langle Ay - Az, y - z \rangle \geq 0$.

By max. mono. of A:

We have $x^* = A(\lambda X + (1-\lambda)y)$. $\Rightarrow \lambda x + (1-\lambda)y \in \mathbb{L}(x^*)$.

Closedness:

By demi-conti of A. $z_n \in \mathbb{L}(x^*) \rightarrow z$.

$\Longrightarrow Az_n \equiv x^* \longrightarrow Az$, So: $Az = x^*$. $z \in \mathbb{L}(x^*)$.

By Mazur's Thm. $\mathbb{L}(x^*)$ is also weakly closed.