Application of Mono. Op. Theory

(1) p - Laplace:

Recall $\Omega \subseteq \mathbb{R}^n$ bdd domain, $p > 1$.

$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ in Ω, $u = 0$ on $\partial\Omega$

procedure:

i) Assume enough regularity, i.e. assume:

$u : \bar{\Omega} \to \mathbb{R}'$ exists & suff. regular.

ii) Multiply with test func.:

$$\int_\Omega f \varphi \, dx = -\int_\Omega \operatorname{div}(|\nabla u|^{p-2}\nabla u)\varphi \, dx, \quad \varphi \in C_c^\infty.$$

iii) Get rid of regularity via integrate by part:

$$\begin{aligned}
RHS &= -\sum_1^n \int_\Omega \partial_i(|\nabla u|^{p-2}\partial_i u)\varphi \, dx \\
&= \sum_1^n \int_\Omega |\nabla u|^{p-2}\partial_i u \, \partial_i \varphi \, dx - \int_{\partial\Omega} |\nabla u|^{p-2}\partial_i u \, \varphi \cdot \nu^i \, dx. \\
&= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx.
\end{aligned}$$

iv) Identify energy space: Set $\varphi = u$.

$$\|\nabla u\|^p_{L^p(\Omega)} = \int_\Omega |\nabla u|^p = \int f u \, dx$$

$$\le \|f\|_{p'} \|u\|_p \underset{\text{zero-trace}}{\overset{\text{Poincaré}}{\le}} C_p \|f\|_{p'} \|\nabla u\|_p$$

$$\Rightarrow \|\nabla u\|_p \le C_p^{\frac{1}{p-1}} \|f\|_{p'}^{\frac{1}{p-1}}.$$

Apply Poincaré inequi. again: $\|u\|_{W_0^{1,p}} \le \tilde{C}_p \|f\|_{p'}^{\frac{1}{p-1}}$.

ii) Identify test function space:

$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int f \varphi \, dx$. Since $|\nabla u|^{p-2} \nabla u \in (L^{p'}(\Omega))^d$. So: $\nabla \varphi \in (L^p(\Omega))^d$.

i.e. Test func space $= \overline{C_c^\infty(\Omega)}^{\|\nabla(\cdot)\|_{L^p}} = W_0^{1,p}(\Omega)$

<u>Def</u>: $f^* \in (W_0^{1,p}(\Omega))^*$. $u \in W_0^{1,p}(\Omega)$ is weak sol. for p-Laplace equation if $\forall v \in W_0^{1,p}(\Omega)$

$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \langle f^*, v \rangle_{W_0^{1,p}}$ holds.

<u>Lemma</u>. i) $(W^{1,p}(\Omega))^* = L^{p'}(\Omega) + \operatorname{div}((L^{p'}(\Omega))^d)$

$$\overset{\Delta}{=} \{ (v \mapsto \int_\Omega f v \, dx + \int_\Omega F \cdot \nabla v) \mid f \in L^{p'}(\Omega), F \in (L^{p'}(\Omega))^d \}.$$

$$(W_0^{1,p}(\Omega))^* = \operatorname{div}((L^{p'}(\Omega))^d).$$

$$\overset{\Delta}{=} \{ (v \mapsto \int_\Omega F \cdot \nabla v \, dx) \mid F \in (L^{p'}(\Omega))^d \}.$$

iii) $W_0^{1,p}$, $W^{1,p}$ are separa. Banach if $p \geq 1$. reflex if $p > 1$

Thm. (Weak - solvability)

$A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ defined by :

$\langle Au, v \rangle_{W_0^{1,p}(\Omega)} := \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx$. is

well-def. bdd^. conti. strictly mono. & coercive. bijective. So : A^{-1} is strictly mono. bdd & hemiconti

Rmk : So the weak solution exists uniquely And the system is stable & well-posed

Pf : i) bdd^ & well-def :

$$|\langle Au, v \rangle| \leq \int |\nabla u|^{p-1} |\nabla v| \, dx$$

$$\leq \| |\nabla u|^{p-1} \|_{p'} \cdot \| \nabla v \|_p$$

$$\leq \| \nabla u \|_p^{p-1} \| v \|_{W_0^{1,p}} . \ \forall v .$$

^ : We proved here is A{bdd set} is bdd set. But it's enough.

So : $\| Au \|_{(W_0^{1,p})^*} = \| u \|_{W_0^{1,p}}^{p-1}$.

ii) Continuity :

For $u_n \to u$ in $W_0^{1,p}(\Omega)$. Then $\nabla u_n \to \nabla u$ in $(L^p)^d$. So : $\exists (u_{n_k})$ st. $\nabla u_{n_k} \to \nabla u$. a.s.

With $t \mapsto |t|^{p-2}t$ is conti.

So : $|\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \to |\nabla u|^{p-2}\nabla u$. a.s.

Note that $(|\nabla u_n|^{p-2}\nabla u_n)$ is uniformly $L^{p'}$-integrable. Since $\nabla u_n \xrightarrow{L^p} \nabla u$. a.e.

So : $|\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \xrightarrow{L^{p'}} |\nabla u|^{p-2}\nabla u$

By subseq convergence principle : it holds for the whole seq.

$$\|Au - Av\|_{W^{-1,p'}} \leq \||\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\|_{p'} \to 0.$$ So A is conti.

3') Strictly mono :

$0 = \langle Au - Av, u - v\rangle \iff (f(u) - f(v))\cdot(u-v) = 0$

(Integrand is nonnegative)

where $f(t) = |t|^{p-2}t$ is also strictly mono. So : it equi. with $\nabla u = \nabla v$.

$\overset{\text{Poincaré}}{\Longrightarrow}$ $0 \leq \|u - v\|_{L^p} \leq \|\nabla u - \nabla v\|_{L^p} = 0$.

4') Coercivity :

$$\langle Au, u\rangle_{W_0^{1,p}} = \int_\Omega |\nabla u|^p \overset{\text{Poincaré}}{\geq} \tilde{C}_p \|u\|^p_{W^{1,p}}$$

So, we can apply Browder - Minty's Thm.

(2) p - Stokes equation:

For $A, B \in \mathbb{R}^{d\times d}$, we denote $A : B \triangleq \sum_{i,j} a_{ij} b_{ij}$

And for $p > 1$, $v : \Omega \to \mathbb{R}^d$.

Consider on bdd Lip domain $\Omega \subseteq \mathbb{R}^d$.

$-\operatorname{div}(S(Dv)) + \nabla \pi = f$ on Ω, $\operatorname{div} v = 0$ in Ω

$v = 0$ on $\partial\Omega$. where $S(A) := \nu_0 (\delta + |A|)^{p-2} A$.

process as before:

$\int_\Omega -\operatorname{div}(S(Dv)) \varphi + \nabla \pi \cdot \varphi =$

$\int_\Omega -\operatorname{div}(S(Dv) - \pi I_{d}) \cdot \varphi =$

$-\sum_{i,j} \int_\Omega \partial_j (S(Dv)_{ij} - \pi \delta_{ij}) \varphi_i =$

$\sum_{i,j} \int_\Omega (S(Dv)_{ij} - \pi \delta_{ij}) \partial_j \varphi_i - \int_{\partial\Omega} \square\, \varphi_i \cdot \nu^j dx$

$= \int_\Omega (S(Dv) - \pi I_d) : \nabla \varphi^T = \int_\Omega \square : \nabla \varphi$.

So: $LHS = \int_\Omega (\cdots) : D\varphi$. by sym of S.

Def: For $f^* \in (W_0^{1,p}(\Omega)^{\otimes d})^*$, $(v, \pi)^T \in W_0^{1,p}(\Omega)^{\otimes d} \times L_0^{p'}$

is weak solution of p - Stokes equation where $L_0^{p'}(\Omega) \triangleq \{\eta \in L^{p'} \mid \int_\Omega \eta = 0\}$. if:

$\int_\Omega S(Dv) : D\varphi\, dx - \int_\Omega z\, \mathrm{div}(\varphi)\, dx = \langle f^*, \varphi \rangle_{V_0^{1,p}}$

$\int_\Omega \eta\, \mathrm{div}(v)\, dx = 0$. holds for $\forall (\varphi, \eta)^T \in (W_0^{1,p}(\Omega))^d \times L^{p'}(\Omega)$.

Rmk: i) Zero mean of z is for uniqueness

since $\int_\Omega (z+c)\, \mathrm{div}(\varphi) = \square + \int_{\partial\Omega} c\varphi^i \cdot v^i$

$= \int_\Omega z\, \mathrm{div}\varphi \Rightarrow z+c$ also solves it!

ii) It's not necessary to get rid of regularity of v in the 2^{nd} equation $\int_\Omega \eta\, \mathrm{div}(v) = 0$ since $v \in W_0^{1,p}(\Omega)^{\otimes d}$ is enough.

iii) The main difficulty is the PDE can't write in $\langle Au, v\rangle$. s.t. A coercive

Lemma (hydro-mechanical formulation)

For $f^* \in (W_0^{1,p}(\Omega)^{\otimes d})^*$. $V \in (W_0^{1,p}(\Omega))^d$.

i) $\exists z \in L_0^{p'}(\Omega)$. s.t. (v, z) is a weak solution of p-Stokes equation.

ii) $v \in V_p := \{ \varphi \in (W_0^{1,p}(\Omega))^d \mid div(\varphi) = 0 \text{ in } \Omega \}$ &

it weakly solves hydromechanical p-Stokes:

$\int S(Dv) : D\varphi \, dx = \langle f^*, \varphi \rangle_{W_0^{1,p}(\Omega)^d}$ for $\forall \varphi \in V_p$

We have : i) $\Longleftrightarrow$ ii)

__Rmk:__ $(V_p, \|\cdot\|_{W_0^{1,p}})$ is cls of $W_0^{1,p}(\Omega)$. In case of hydro-mech. we restrict $\varphi \in V_p \subsetneq W_0^{1,p}$.

__Pf:__ __Lem.__ (de Rham)

If $f^* \in (W_0^{1,p}(\Omega)^{\otimes d})^*$ satisfies $\langle f^*, \varphi \rangle = 0$, for $\forall \varphi \in V_p$. Then : $\exists \lambda \in L_0^{p'}(\Omega)$.

s.t. $\langle f^*, \varphi \rangle_{W_0^{1,p}} = \int \lambda \, div(\varphi)$. $\forall \varphi \in W_0^{1,p}(\Omega)^{\otimes d}$

__Rmk:__ Set $\nabla : L_0^{p'}(\Omega) \to (V_p)^0 := \{ f^* \in (W_0^{1,p}(\Omega)^d)^* : \langle f^*, \varphi \rangle_{W_0^{1,p}} = 0 . \ \forall \varphi \in V_p \}$. annihilator of V_p. Def by $\nabla \lambda = f^*$. on above

$\Rightarrow$ We have $L^{p'}(\Omega) \underset{iso}{\overset{\nabla}{\cong}} (V_p)^0$.

i) $\Rightarrow$ ii) Since $\int \eta \, div(v) = 0$. for $\forall \eta \in C_c^\infty$

$\Rightarrow v \in V_p$.

Also let $\varphi \in V_p$. $\int S(Dv) : D\varphi = \square$ holds.

ii) $\Rightarrow$ i) $\phi = (\varphi \mapsto \langle f^*, v \rangle_{W_0^{1,p}} - \int S(Dv) : D(\varphi)) \in (V_p)^\circ$

Apply de-Rham Lemma:

$\exists\, z \in L_0^{p'}(\Omega)$ s.t. $-\int z \,\mathrm{div}(\varphi) = \phi(\varphi)$

$= \langle f^*, v \rangle - \int S(Dv) : D\varphi$ for $\forall \varphi \in W_0^{1,1}(\Omega)^d$

Thm (Well-posedness of hydro-mechanical form)

$\hat{S} : V_p \to V_p^*$ is defined by $\langle \hat{S} v, \varphi \rangle_{V_p} \stackrel{\Delta}{=}$ $\int_\Omega S(Dv) : D\varphi \, dx$ for $\forall v, \varphi \in V_p$. Then:

$\hat{S}$ is well-def, bdd, conti, strictly mono, coercive, bijective; $\hat{S}^{-1}$ is bdd, strictly mono and demi-conti. ($f_n^* \to f^*$ in $V_p^* \Rightarrow$ $v_n \rightharpoonup v$ in V_p)

Pf: 1) bdd & well-def:

$$|\langle \hat{S} v, \phi \rangle_{V_p}| = |\int_\Omega S(Dv) : D\phi \, dx|$$

$$\stackrel{\text{Hölder}}{\le} \nu_0 \,\|(\delta + |Dv|)^{p-1}\|_{p'} \|D\phi\|_p$$

$$= \nu_0 \,\|\delta + |Dv|\|_p^{p-1} \|D\phi\|_p$$

$$\le \nu_0 \,(\delta\, |\Omega|^{\frac{1}{p}} + \|v\|_{V_p})^{p-1} \|\phi\|_{V_p}$$

2) Continuity:

$v_n \to v$ in $V_p \Rightarrow \exists (n_k),\ v_{n_k} \to v$ a.e.

$\overset{S\text{ conti}}{\Rightarrow} S(DV_{nk}) \to S(DV)$. a.e.

And (DV_k) is L^p-u.i. $\Rightarrow S(DV_k)$ is $L^{p'}$-u.i.

So: $S(DV_{nk}) \xrightarrow{L^{p'}} S(DV)$.

With subseq convergence argument.

3) Strict mono.:

$0 = \langle \hat{S}v - \hat{S}\varphi, v - \varphi \rangle.$ $\overset{(\text{integrand} \geq 0)}{\Leftrightarrow}$

$(S(Dv) - S(D\varphi)) \cdot (Dv - D\varphi) = 0$. a.e

S is strict mono. So it equi. $Dv = D\varphi$. a.e.

By Korn's inequ.: $\|\nabla u\|_p = C_K \|Du\|_p$

So: $v - \varphi \equiv$ const. $\overset{\text{boundary}}{\underset{\text{cond.}}{\Rightarrow}}$ $v = \varphi$.

Rmk: Korn's inequai. only works for $p > 1$.

But Poincare trace inequi. works for $p \geq 1$.

4) Coercivity:

$\langle \hat{S}v, v \rangle_{V_p} = \nu_0 \int_\Omega (\delta + |Dv|)^{p-2} |Dv|^2 dx.$

If $p > 2$. LHS $\geq \nu_0 \int_\Omega |Dv|^p dx$

If $1 < p < 2$. $(\delta + |Dv|)^{p-2} |Dv|^2 \overset{\text{AM-GM}}{\geq}$

$\frac{1}{2}(\delta + |Dv|)^p - \delta^2 (\delta + |Dv|)^{p-2}$

$= \frac{1}{2}|DV|^p - \delta^p / (1+|DV|/\delta)^{2-p}$

$\geq \frac{1}{2}|DV|^p - \delta^p$

Then apply Korn's & Poincaré inequali. again.

<u>Cor.</u> (well-posed for p-Stokes)

$\forall f^* \in ((W_0^{1,p}(\Omega))^d)^*$. Then $\exists$ unique weak solution $(v,\pi) \in (W_0^{1,p}(\Omega))^d \times L_0^{p'}(\Omega)$ for p-Stokes equation s.t. it depends on f^* semicontinuously. i.e. $(f_n^*) \subset ((W_0^{1,p}(\Omega))^d)^*$ $\to f^*$. $\Rightarrow (v_n, \pi_n) \subset (W_0^{1,p}(\Omega))^d \times L_0^{p'}(\Omega)$ $\to (v,\pi)$. for those corrspd. sol.'s.

<u>Lemma</u>. For $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, Lip. domain & $\forall p \in (1,\infty)$. $\exists B : L_0^p(\Omega) \to W_0^{1,p}(\Omega)$ which is called Bogovski operator. s.t. $\mathrm{Div} \circ B = \mathrm{id}_{L_0^p}$ (Div is divergence in weak derivative sense for $W_0^{1,p}(\Omega)$)

<u>Pf of Cor.</u>:

By hydromechanical lem.: weak solution for p-Stokes exists for $\forall f^* \in (W_0^{1,p}(\Omega)^{\otimes d})^*$.

Let $i\lambda_{V_p} : V_p \to (W_0^{1,p}(\Omega))^{\otimes d}$. $\Rightarrow (i\lambda_{V_n})^* f^* \in V_p^*$

So $\exists v \in V_p$. $\int S(Dv) : D\varphi = \langle (i\lambda_{V_n})^* f^*, \varphi \rangle_{V_p}$

$= \langle f^*, \varphi \rangle_{W_0^{1,p}}$. $\forall \varphi \in V_p$.

If $(v, \tilde{z})$ also solve equation. Then:

$\int_\Omega \tilde{z}\, Div(\varphi) = \int_\Omega z\, Div(\varphi)$. $\forall \varphi \in W_0^{1,p}(\Omega)$.

By Lemma. $\int_\Omega (\tilde{z} - z)^2 = 0$. $\Rightarrow \tilde{z} = z$. a.e.

If $f_n^* \xrightarrow{(W_0^{1,p})^*} f^*$. look in hydromech. eq. ($(i\lambda_{V_n})^*$ conti.)

$\Rightarrow v_n \xrightarrow{V_p} v$. And $\exists (z_n)$. z with (v_n). v

solves it. Since $\sup_{W_0^{1,p}} \langle z_n, div(\varphi) \rangle \overset{\text{Lem.}}{=} \sup_{L^p}$

$\langle z_n, \tilde{\varphi} - \int_\Omega \tilde{\varphi} \rangle = \sup_{L^p} \langle z_n, \tilde{\varphi} \rangle \overset{p\text{-Stokes}}{\leq} \square < \infty$

from p-Stokes equation and z_n is zero-mean

So: $\|z_n\|_{L^{p'}} \leq C < \infty$. $\exists (n_k)$. s.t. $z_{n_k} \rightharpoonup \tilde{z}$.

Since $L^{p'}$ is reflexive.

But p-Stokes equation has unique sol.

So: $\tilde{z} = z$. With subseq convergence prin.

$\Rightarrow z_n \rightharpoonup z$. So: $(v_n, z_n) \rightharpoonup (v, z)$ in

$W_0^{1,p} \times L_0^{p'}$.