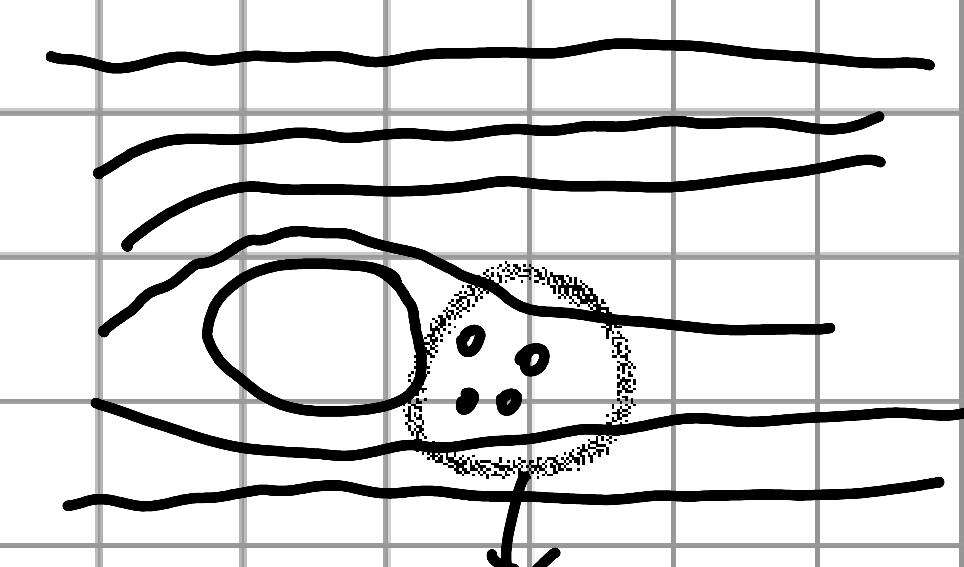
Pseudo-Mono. Operator

· For general p-Navier-Stokes equation:

$\mathrm{div}(v\otimes v) + \nabla\pi - \mathrm{div}(S(Dv)) = f$ in Ω.

Because of the existence of $\mathrm{div}(v\otimes v)$. Minty's Thm for mono. operator doesn't work since it won't lead to mono. operator.



from $\mathrm{div}(v\otimes v)$

($\langle Av, \varphi\rangle := -\int_\Omega v\otimes v : D\varphi$ isn't mono. $v, \varphi\in V_p$)

· Recall the IFT in $\mathbb{R}^1$, if we drop "strict mono." cond. i.e. assume $A:\mathbb{R}^1\longrightarrow\mathbb{R}^1$ is conti. & coercive. Then: we can only get A is surjective rather bijective.

Next, we will also develop generalization of IFT in Banach space as Minty's.

(1) <u>Condition(M)</u>:

We first want to replace the "conti" condition by condition(M).

Def: $(X, \|\cdot\|_X)$ is real Banach space.

$A: X \to X^*$ is type (m) if $\forall (x_n) \subset X$

$x_n \rightharpoonup x \in X$

$Ax_n \rightharpoonup x^* \in X^*$ $\Rightarrow$ $Ax = x^*$.

$\overline{\lim\limits_{n\to\infty}} \langle Ax_n, x_n \rangle \leq \langle x^*, x \rangle$

Prop. For $\dim X < \infty$. Then: $A: X \to X^*$ is conti.

$\Leftrightarrow$ A is type (m)/pseudo-mono. & locally bdd

Pf: ($\Rightarrow$) is trivial from $\dim X$ finite.

($\Leftarrow$) $\forall x_n \to x$. Since (Ax_n) is bdd

$\exists (Ax_{n_k}) \to x^*$. $\Rightarrow$ $\langle Ax_{n_k}, x_{n_k} \rangle \to \square$

So: $Ax = x^*$ from type (m).

Lemma. i) A mono. radially conti. $\Rightarrow$ type (m)

ii) A weakly / strongly conti. $\Rightarrow$ type (m)

iii) A type (m). B strongly conti. $\Rightarrow$

$A+B$ is type (m).

iv) A type (m). B mono. weakly conti. $\Rightarrow$

$A+B$ is type (m).

v) A type (m). locally bdd + X reflexive $\Rightarrow$

A demi-conti.

Pf: i) is from Minty's trick

ii) is from definition

iii) Assume $(x_n) \rightharpoonup x$. $\square$. $\square$.

Note $Bx_n \to Bx$ by strong conti.

$\Rightarrow Ax_n \rightharpoonup x^* - Bx$. $\lim \langle Ax_n, x_n \rangle \leq \square$

So: $Ax = x^* - Bx$ by type (m) of A.

iv) Assume $\square$. Note $Bx_n \rightharpoonup Bx$ from B is weakly conti. $\Rightarrow Ax_n \rightharpoonup x^* - Bx$

$\langle (A+B)x_n, x_n \rangle \overset{\text{mono.}}{\geq} \langle Ax_n, x_n \rangle + \langle Bx_n, x \rangle$

$+ \langle Bx, x_n \rangle - \langle Bx, x \rangle$

$\Rightarrow \overline{\lim_{n\to\infty}} \langle Ax_n, x_n \rangle \leq \langle x^* - Bx, x \rangle$

So: $x^* - Bx = Ax$

v) For $x_n \to x$. $\overset{\text{local bdd}}{\Longrightarrow} (Ax_n)$ is bdd

By reflexive of X, $\exists (n_k)$. s.t.

$Ax_{n_k} \rightharpoonup x^*$. So: $\langle Ax_{n_k}, x_{n_k} \rangle \to \square$

So $Ax = x^*$. by type (m).

With subseq convergence argument.

e.g. (type (m) + type (m) $\not\Rightarrow$ type (m))

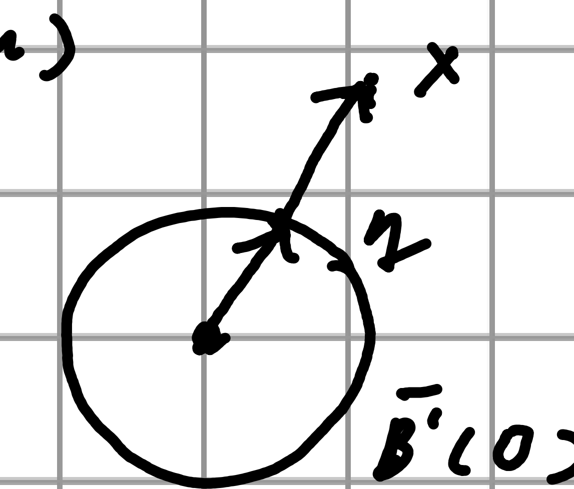
$(H, (\cdot,\cdot)_H)$ is Hilbert space. with o.n.b.

$(e_n)_{\mathbb{Z}^+}$. $R_H : H \to H^*$ is Riesz isomorphism.

i.e. $\langle R_H x, y \rangle = (x, y)_H$ for $\forall x, y \in H$.

1) Set $A = -R_H$. weakly conti. $\Rightarrow$ type(M)

2) $Bx := R_H \left(\underset{y \in \bar{B}_1(0)}{\operatorname{argmin}} \|x - y\|_H^2 \right)$



<u>Rmk</u>: 2)'s equi. $(R_H^{-1} Bx - x, R_H^{-1} Bx - y)_H \leq 0$.

for $\forall y \in \bar{B}_1(0)$.

So B is mono. Lipschitz conti.:

$$\langle Bx - By, x - y \rangle = (R_H^{-1} Bx - R_H^{-1} By, x - y)_H$$

$$= \|R_H^{-1} Bx - R_H^{-1} By\|^2 + (R_H^{-1} Bx - R_H^{-1} By, x - R_H^{-1} Bx) + \square.$$

$$\overset{\text{Rmk}}{\geq} \|R_H^{-1} Bx - R_H^{-1} By\|^2 = \|Bx - By\|^2.$$

By Lemma i) above. B is type(M).

Choose $x_n := e_1 + e_n \xrightarrow{n \to \infty} e_1$. And note:

$$(A+B)x_n = R_H \left(\frac{e_1 + e_n}{\|e_1 + e_n\|} - e_1 - e_n \right) \rightharpoonup \left(\frac{1}{\sqrt{2}} - 1 \right) R_H e_1$$

Since $Bx_n = R_H \left(\frac{e_1 + e_n}{\|e_1 + e_n\|} \right) = R_H (e_1 + e_n) / \sqrt{2}$

Also $\overline{\lim} \langle (A+B)x_n, x_n \rangle < \langle \square, e_1 \rangle$.

But $(A+B)e_1 = -Re_1 + Re_1 = 0 \neq (\frac{1}{\sqrt{2}}-1)Re_1$

So $A+B$ isn't type(m)!

(2) <u>Pseudo-Monotone:</u>

Note type(m) func. isn't stable under summation. But pseudo-mono can, which is intermediate between radially conti. + mon. and type(m) operator.

<u>Def:</u> $A: X \to X^*$ is pseudo-mono. if $\forall x_n \rightharpoonup x$ satisfies $\overline{\lim} \langle Ax_n, x_n - x\rangle \le 0 \Rightarrow \forall y \in X$. we have $\langle Ax, x-y\rangle \le \underline{\lim} \langle Ax_n, x_n - y\rangle$.

<u>Rmk:</u> i) Pseudo-mono is actually notion of continuity when treating some variation inequi.

ii) Note let $y = x$, Then:

$$0 = \langle Ax, x-x\rangle \le \underline{\lim} \langle Ax_n, x_n - x\rangle \le 0$$

$$\Rightarrow \lim \langle Ax_n, x_n - x\rangle = 0.$$

So: $\langle Ax, x-y\rangle \le \underline{\lim} \langle Ax_n, x_n - y\rangle$

$= \underline{\lim} \langle Ax_n, x - y\rangle$

If $Ax_n \to x^*$. Then $\forall y \in X$

We have $\langle Ax, x-y \rangle \le \langle x^*, x-y \rangle$.

let $y = x \pm z$. $\Rightarrow \langle Ax, z \rangle = \langle x^*, z \rangle$.

So: $Ax = x^*$.

Lemma i) mono + radial conti. $\Rightarrow$ pseudo-mono.

$\Rightarrow$ type (M).

ii) Strongly conti. $\Rightarrow$ pseudo-mono.

iii) A, B pseudomono. $\Rightarrow$ $A+B$ pseudo-mono.

iv) A pseudo-mono, locally bdd and X is reflexive $\Rightarrow$ A is hemi-conti.

Pf: i) i) Assume $x_n \rightharpoonup x$. $\overline{\lim} \langle Ax_n, x_n - x \rangle \le 0$

Set $y_t = (1-t)x + ty$, By mono.:

$\langle Ay_t - Ax, y_t - x \rangle \ge 0$. i.e.

$t \langle Ax_n, x_n - y \rangle \ge -(1-t)\langle Ax_n, x_n - x \rangle + \square$

divide t on both sides. let

$n \to \infty$: $\underline{\lim} \langle Ax_n, x_n - y \rangle \ge \langle Ay_t, x-y \rangle$

(Since $\lim \langle Ax_n, x_n - x \rangle = 0$)

Set $t \searrow 0$. with radial conti.

We have: $\underline{\lim}\, \square \ge \langle Ax, x-y \rangle$.

2) Assume $X_n \rightharpoonup X$. $AX_n \rightharpoonup X^*$. □

From Rmk ii) above. We only need that:

$\overline{\lim} \langle AX_n, X_n - x \rangle \le 0$. Since:

$LHS = \overline{\lim} \langle AX_n, X_n \rangle - \lim \langle AX_n, x \rangle$

$\le \langle X^*, X \rangle - \langle X^*, X \rangle = 0$.

ii) from $AX_n \to Ax$, $X_n \rightharpoonup X \Rightarrow \langle AX_n, X_n \rangle \to$ □

iii) Assume $X_n \rightharpoonup X$. $\overline{\lim} \langle (A+B)X_n, \cdot\cdot \rangle \le 0$

Set $a_n = \langle AX_n, X_n - X \rangle$. $b_n = \langle BX_n, X_n - x \rangle$.

Claim: $\overline{\lim}\, a_n \le 0$. $\overline{\lim}\, b_n \le 0$.

By contradiction, if $\overline{\lim}\, a_n = a > 0$.

Then: $\overline{\lim}\, b_{n_k} \le \overline{\lim} (a_{n_k} + b_{n_k}) - \lim a_{n_k} < 0$

where $a_{n_k} \to a$.

But by pseudo-mono of B: $\underline{\lim}\, b_{n_k} \ge 0$

after we choose $y = x$. $\Rightarrow$ Contradict!

iv) Pseudomono $\Rightarrow$ type (M). with Lem. before

Rmk: Pseudo-mono. is actually a strict intermediate class in i).

e.g., i) (type(M) $\not\Rightarrow$ pseudo-mono.)

Even weakly conti. $\not\Rightarrow$ pseudo-mono.

Set $(X, (\cdot)_X)$ is Hilbert space with o.n.b. (e_i). $AX = -R_X x$. Note that $x_n = e_n \rightharpoonup 0$. $\lim \langle Ax_n, x_n - 0\rangle = -1 < 0$.

But let $y = 0$. We have:

$\langle A0, 0-0\rangle = 0 > -1 = \lim \langle Ax_n, x_n - 0\rangle$

ii) (pseudo-mono. $\not\Rightarrow$ radially conti. + mono.)

For $p \geq \frac{3d}{d+2}$. $\Omega \underset{\text{bdd}}{\subseteq} \mathbb{R}^d$. $d \geq 2$.

$C : V_p \to V_p^*$. $\langle Cv, \varphi\rangle := -\int_\Omega v \otimes v : D\varphi \, dx$ is bdd. pseudomonote. (proved below)

But it's not monotone. It follows from cancelling prop. Set tu, tv. Let t large enough: $\square = -t^2 \langle Cu, v\rangle - t\langle Cu, v\rangle < 0$.

Thm. (Main Theorem on Pseudo-mono.)

$(X, \|\cdot\|_X)$ is real separ. reflexive Banach

If $A : X \to X^*$ is bdd*, pseudo-mono. and coercive. Then: A is surjective.

Rmk: It holds if drop separa. and replace pseudomono by typeI(M) & bdd by locally bdd*

Pf: Fix $x^* \in X^*$. Show: $\exists x \in X$. s.t. $\forall y \in X$. We have: $\langle Ax, y \rangle = \langle x^*, y \rangle$.

Using Galerkin method: $X_n = \mathrm{span}\{e_k\}_1^n$, where $(e_k)_k$ is o.n.b.

And prove: $\exists z_n \in X_n$. s.t. $\langle A z_n, y_n \rangle = \langle X_n^*, y_n \rangle$. $\forall y_n \in X_n$. where $X_n^* = (i_{X_n})^* X$.

1) Well-posed: (existence for $\forall n$)

As before, we only need A is coercive and hemi-conti. follow from lem.

2) Stability:

bdd of solutions seq is from coercive and the image seq. is also bdd since A is bdd*. (As before, can be proved by local bdd)

3) Weak convergence:

If (z_n) is the solution seq. By reflexive: $\exists z_{n_k} \rightharpoonup z$. $A z_{n_k} \rightharpoonup \xi^*$.

Note $\forall y \in X$. $\exists y_n \in X_n \to y$. Then:

$$\langle y^*, \eta \rangle = \lim_{n\to\infty} \langle y^*, y_{n_k} \rangle$$

$$= \lim_{n\to\infty} \langle A z_{n_k}, y_{n_k} \rangle$$

$$= \lim_{n\to\infty} \langle x_{n_k}^*, y_{n_k} \rangle = \langle x^*, \eta \rangle$$

So: $y^* = x^*$. $\overset{\text{As before}}{\Longrightarrow}$ $\overline{\lim} \langle A z_{n_k}, z_{n_k} \rangle = \langle x^*, z \rangle$

With A is type (M). $\Rightarrow$ $Az = x^*$.

(3) <u>Application</u>:

Next we consider p-NS equation:

$$-\operatorname{div}(S(Dv)) + \operatorname{div}(v \otimes v) + \nabla \pi = f \quad \text{in } \Omega$$

$$\operatorname{div}(v) = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

where $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$. bdd Lip domain occupied by fluid and external force $f: \Omega \to \mathbb{R}^d$.

<u>Def</u>: For $p \geq 3d/(d+2)$. $f^* \in (W_0^{1,p}(\Omega)^d)^*$. (v, π) $\in W_0^{1,p}(\Omega)^d \times L_0^{p'}(\Omega)$ is weak solution for p-NS equation if

$$\int_\Omega (S(Dv) - v \otimes v) : D\varphi - \pi \operatorname{div}(\varphi)\, dx = \langle f^*, \varphi \rangle$$

$$\int_\Omega \eta \operatorname{div}(v) = 0. \quad \text{for } \forall\ (\varphi, \eta) \in W_0^{1,p} \times L^{p'}.$$

Lemma. $v \in W_0^{1,p}(\Omega)^d$. Then i). ii) equi. :

i) $\exists\ z \in L_0^{p'}(\Omega)$. st. (v, z) is weak solution for p-NS equation.

ii) $v \in V_p$. v is weak solution of hydromechanical p-NS equation. i.e.

$$\int_\Omega (S(Dv) - v \otimes v) : D\varphi = \langle f^*, \varphi \rangle$$

for $\forall \varphi \in V_p$.

Pf: identical with before.

Lemma (Prop of weak convective term)

$C : V_p \to V_p^*$ defined by $\langle Cv, \varphi \rangle :=$
$-\int_\Omega v \otimes v : D\varphi\, dx$ for $v. \varphi \in V_p$. is bdd, pseudo-mono. and has cancelling prop: $\langle Cv, v \rangle_{V_p} = 1$. $\forall v \in V_p$.

Besides, if $p > 3d/_{d+2}$ in additional. then C is strongly conti.

Pf: i) Recall Sobolev embedding:

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega). \text{ where}$$

$$p^* = \begin{cases} dp/d-p & \text{if } p \in [1,d) \\ \forall [1,\infty) & \text{if } p = d \\ \infty & \text{if } p > d. \end{cases}$$ Sobolev conjugate

And note that $p^* \geq 2p' \iff p \geq 3d/d+2$.

So: $\|v\|_{L^{2p'}} \leq C\|v\|_{L^{p^*}} \leq C\|v\|_{V_p}$.

Using $|v \otimes v|^2 = |v|^4$. $|D\varphi| \leq |\nabla \varphi|$. :

$$|\langle Cv, \varphi\rangle| \leq \|v\|^2_{L^{2p'}} \|\varphi\|_{L^p} \leq C\|v\|^2_{V_p} \|\varphi\|_{L^p}$$

2) Cancelling property:

$$\langle Cv, v\rangle \overset{\text{sym}}{=} -\int v \otimes v : \nabla v$$

$$= -\int \sum_{ij} v_i v_j \partial_j v_i$$

$$\underset{\text{by part}}{\overset{\text{integrate}}{=}} \int \sum_{ij} \partial_j (v_i v_j) v_i$$

$$= \int v \otimes v : Dv + \operatorname{div}(v)|v|^2$$

$$\overset{v \in V_p}{=} -\langle Cv, v\rangle$$

3) Pseudo-mono. :

For $v_n \rightharpoonup v$ in V_p, $\overline{\lim} \langle Cv_n, v_n - v\rangle \leq 0$

<u>Lemma</u> (Rellich & Kondrachov)

$W^{k,p}(\Omega) \overset{\text{cpt}}{\hookrightarrow} W^{l,q}(\Omega)$. for $k > l$ and

$k - d/p > l - d/q$. Ω is bdd. $\partial\Omega$ is Lip.

a) We let $q = 2$. ($p^* \geq 2$ since $p = \square$)

(v_n) bdd in $V_p \Rightarrow \exists (v_{n_k}) \xrightarrow{L^2} v$.

And recall $V_p \hookrightarrow L^{2p'}(\Omega)^d \subset L^2(\Omega)^d$

So: $v_n \rightharpoonup v$ in L^2.

(Note it can't imply $v_n \otimes v_n \xrightarrow{L^2} v \otimes v$)

$\Rightarrow (v_n)$ is L^2-bdd

So: $\exists\, v_{n_k} \otimes v_{n_k} \xrightarrow{L^1} v \otimes v$ follows from

$\| v_{n_k} \otimes v_{n_k} - v \otimes v \|_{L^1} \leq \| v_{n_k} \|_{L^2} \| v - v_{n_k} \|_{L^2}$

$+ \| v \|_{L^2} \| v_{n_k} - v \|_{L^2} \to 0 \quad (k \to \infty)$

b) Also. $(v_n \otimes v_n)$ is $L^{p'}$-bdd since

(v_n) is $L^{2p'}$ bdd.

So $\exists\, v_{m_k} \otimes v_{m_k} \xrightarrow{L^{p'}} z$

We can repeat the argument a)

to argue $z = v \otimes v$. by uniqueness

And by subseq convergence principle:

$v_n \otimes v_n \rightharpoonup v \otimes v$ in $L^{p'}$.

i.e. $\langle C v_n, \varphi \rangle_{V_p} \to \langle C v, \varphi \rangle_{V_p}$.

c) By cancelling prop. we see:

$\langle Cv, v-\varphi\rangle = -\langle Cv, \varphi\rangle$

$= \lim -\langle Cv_n, \varphi\rangle = \lim \langle Cv_n, v_n - \varphi\rangle$

4) If $p > 3d/d+2$, then $p^* > 2p'$.

So: $V_p \overset{cpt}{\hookrightarrow} L^{2p'}$.

For $v_n \longrightarrow v$ in V_p. we have:

$\exists (v_{n_k})$. st. $v_{n_k} \otimes v_{n_k} \xrightarrow{L^{p'}} v \otimes v$.

Apply subseq convergence principle.

Thm. (Solvability of hydro-mech. of p-NS)

$A : V_p \to V_p^*$ is defined by: $\forall v, \varphi \in V_p$

$\langle Av, \varphi\rangle_{V_p} = \int_\Omega S(Dv) : D\varphi - v \otimes v : D\varphi \, dx$.

is well-def. bdd, pseudo-mono. coercive

So A is surjective.

Pf: 1) $\langle \hat{S} v, \varphi\rangle = \int_\Omega S(Dv) : D\varphi$ satisfies

the prop. we need. $\overset{Lem}{\Rightarrow}$ A is bdd.

2) pseudo-mono: since $\hat{S}$, C both ✓

3°) Coercive: By cancelling prop.

$\langle Av, v\rangle = \langle \hat{S} v, v\rangle$

Cor. p-NS equation is solvable. (by Lem.)