Calculus of Varia.

Poisson problem:

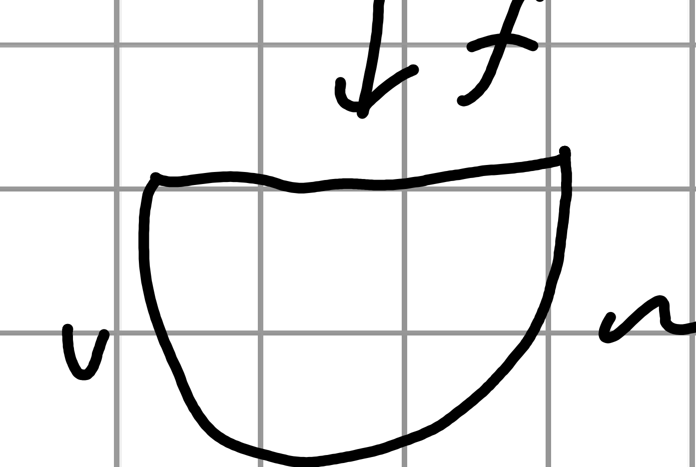
$-\Delta u = f$ on Ω

$u = 0$ on $\partial\Omega$

$\Rightarrow$ Weak form: $\forall v \in W_0^{1,2}(\Omega)$.

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$

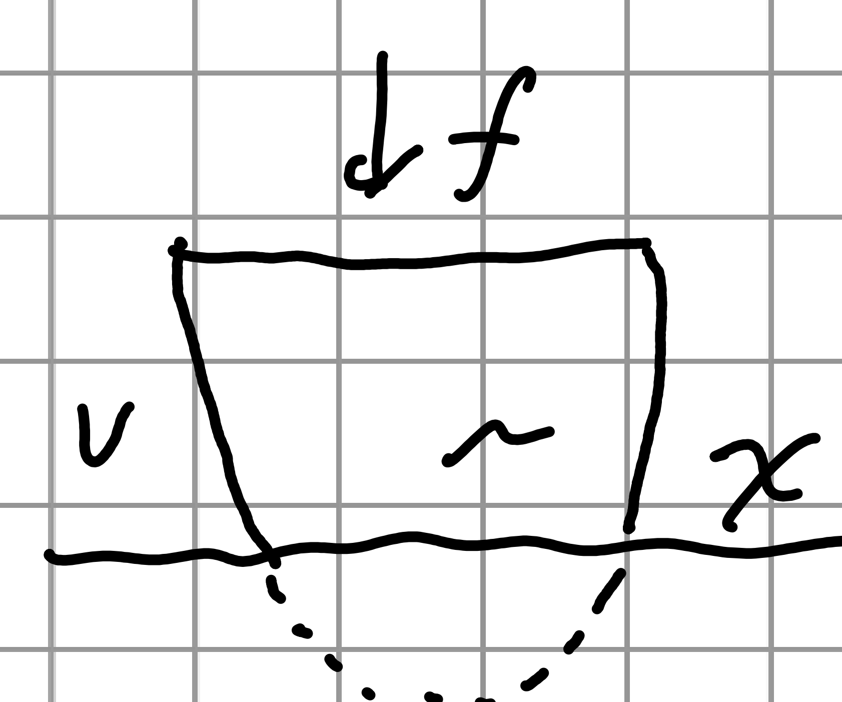
Energy form (let $v = u$):



$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx$$

where $u \in W_0^{1,2}(\Omega)$.

Obstacle Problem:



Strong / weak form: ?

Energy form:

$$I(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 - f u \, dx, & v \geq \chi(x) \text{ on } \Omega. \\ +\infty, & \text{otherwise.} \end{cases}$$

where $u \in W_0^{1,2}(\Omega)$, $\chi \in W^{1,2}(\Omega)$, $\chi \leq 0$ on $\partial\Omega$

(1) **Dirichlet Method:**

Def: $(X, \|\cdot\|_X)$ Banach, $F: X \to \mathbb{R} \cup \{+\infty\}$.

i) F is l.s.c if for $x_n \to x$ in X

$\Rightarrow \underline{\lim}\, F(x_n) \geq F(x)$

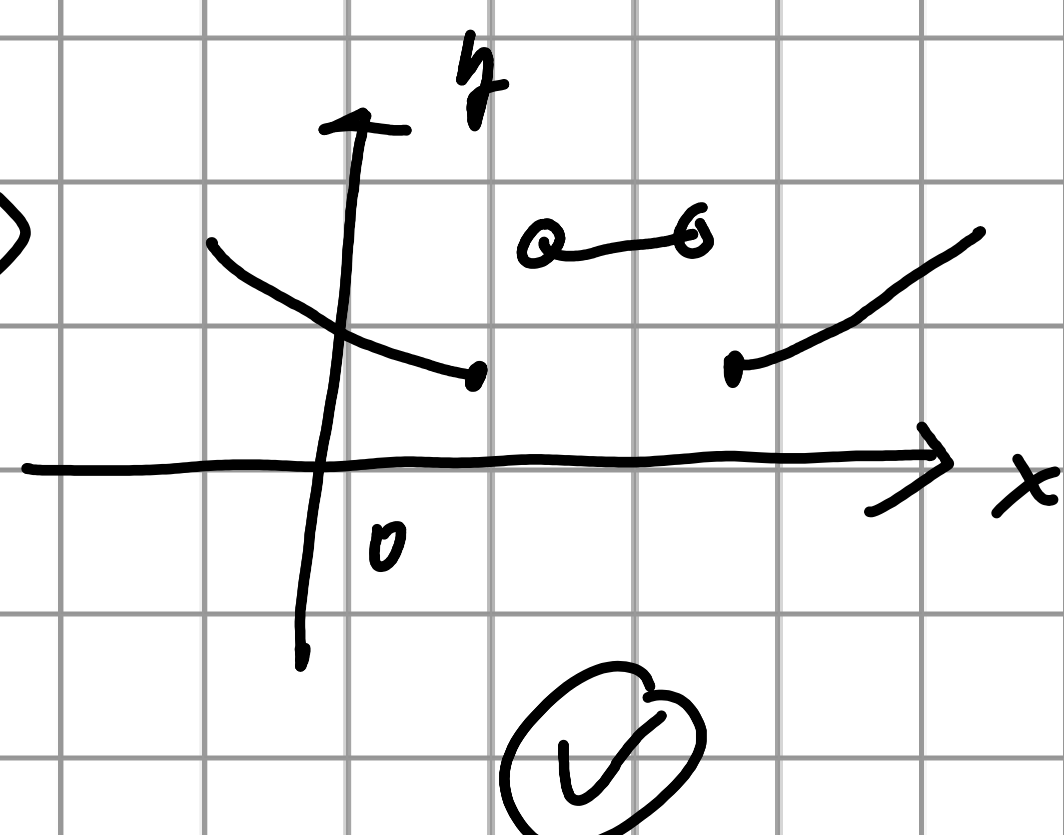
ii) F is weakly l.s.c if for $x_n \rightharpoonup x$.

$\Rightarrow \underline{\lim}\, F(x_n) \geq F(x)$.

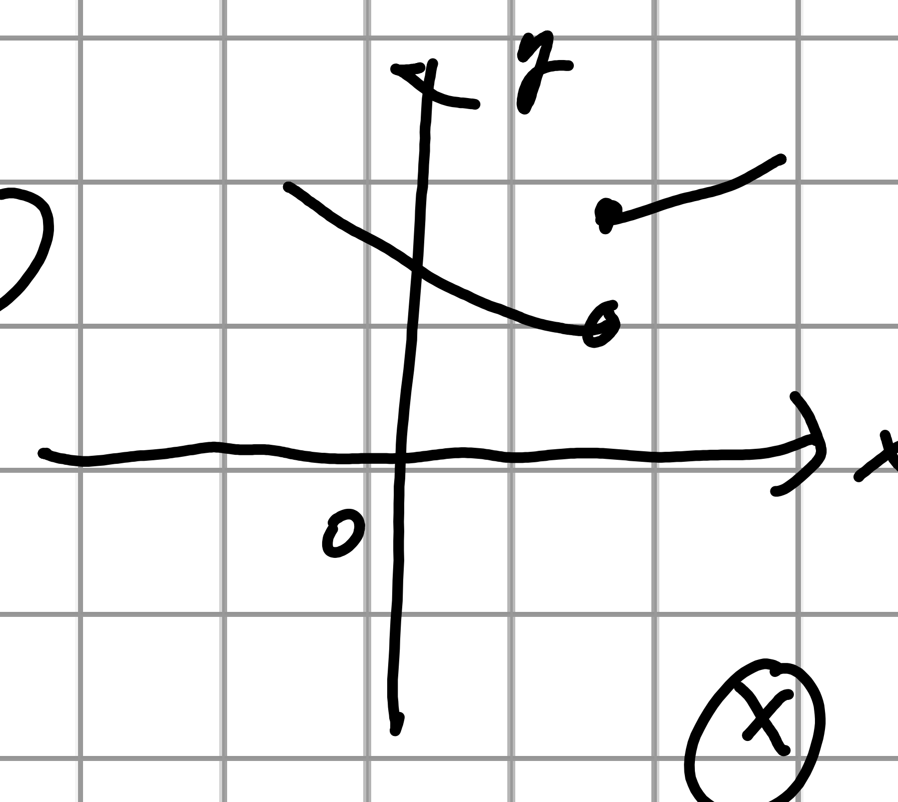
Rmk: i) $\underline{\lim}\, F(x_n) = +\infty$ is allowed

ii) Def means $F(x)$ is lower bdd for each accumulation pt. consistent with alternative Def that $\{F \leq \lambda\}$ is closed. $\forall \lambda \leq +\infty$.

e.g. i)



ii)



Lem. i) w.l.s.c $\Rightarrow$ l.s.c.

ii) l.s.c + convex $\Rightarrow$ w.l.s.c.

Pf: i) is trivial. For ii):

let $(x_n) \rightharpoonup x$ in X and choose subseq (x_{n_k}). s.t. $F(x_{n_k}) \to \underline{\lim}_n F(x_n)$

Apply Mazur's. $\Rightarrow \exists y_k \in \operatorname{conv}\{x_{n_j}\}_{j \geq k}$.

s.t. $y_k = \sum_{j \geq k}^{m_k} \lambda_j^k x_{n_j} \to x$.

So: $F(x) \overset{\text{l.s.c.}}{\leq} \underline{\lim} F(y_n) \overset{\text{convex}}{\leq} \underline{\lim}_k \sum_k^{m_k} \lambda_j^k F(x_{n_j})$

$\leq \underline{\lim}_k \max_{k \leq j \leq m_k} F(x_{n_j})$.

$= \underline{\lim}_k F(x_{n_{k'}}) = \underline{\lim} F(x_n)$

where $k \leq k' \leq m_k$.

e.g. (Indicator)

For $A \subseteq X$. $I_A(x) := \begin{cases} 0 & x \in A \\ +\infty & \text{otherwise}. \end{cases}$

Prop. i) I_A is w.l.s.c $\iff$ A is weakly closed

ii) I_A is l.s.c $\iff$ A is strongly closed

iii) I_A is convex $\iff$ A is convex.

Pf: i) $(\Rightarrow)$ is direct. For $(\Leftarrow)$:

Let $x_n \rightharpoonup x$ in X.

1) if $x \in A$. it's trivial.

2) if $x \notin A$. since A is weakly

closed. so for $(x_{n_k}) \to x$. where

$\underline{\lim} F(x_n) = \lim F(x_{n_k})$. s.t.

$\exists (x_{\tilde{n_k}}) \subset (x_{n_k})$. s.t. $x_{\tilde{n_k}} \notin A$.

(otherwise. $x \in A$. since $(x_{n_k}) \subset A$)

So: $I_A(x) \le +\infty = \underline{\lim} F(x_n)$

$= \lim F(x_{\tilde{n_k}})$.

ii) follows also similar proof of i).

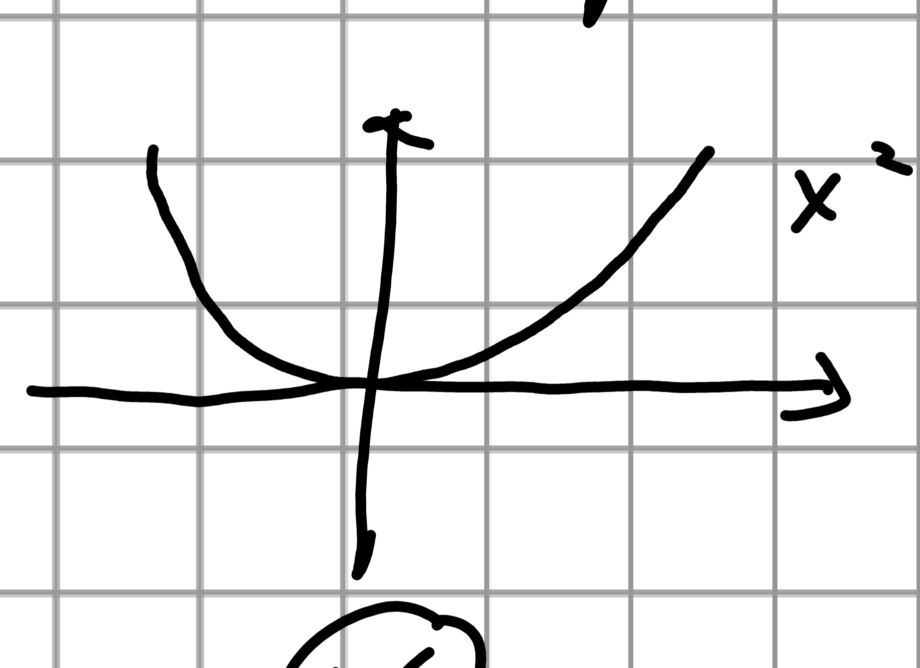
iii) ($\Rightarrow$) is direct. ($\Leftarrow$): consider $x, y \in A$ and case $\exists\, x$ or $y \notin A$.

Def: i) $F: X \to \mathbb{R}^1 \cup \{+\infty\}$ is called proper if the effective domain $\mathrm{dom}(F) := \{x \in X \mid F(x) < \infty\} \neq \emptyset$.

Rmk: I_A is proper $\Longleftrightarrow A \neq \emptyset$.

ii) $F: X \to \mathbb{R}^1 \cup \{+\infty\}$ is called weakly coercive if $F(x) \to +\infty$ as $\|x\| \to +\infty$.

Rmk: It only permits $+\infty$. e.g.



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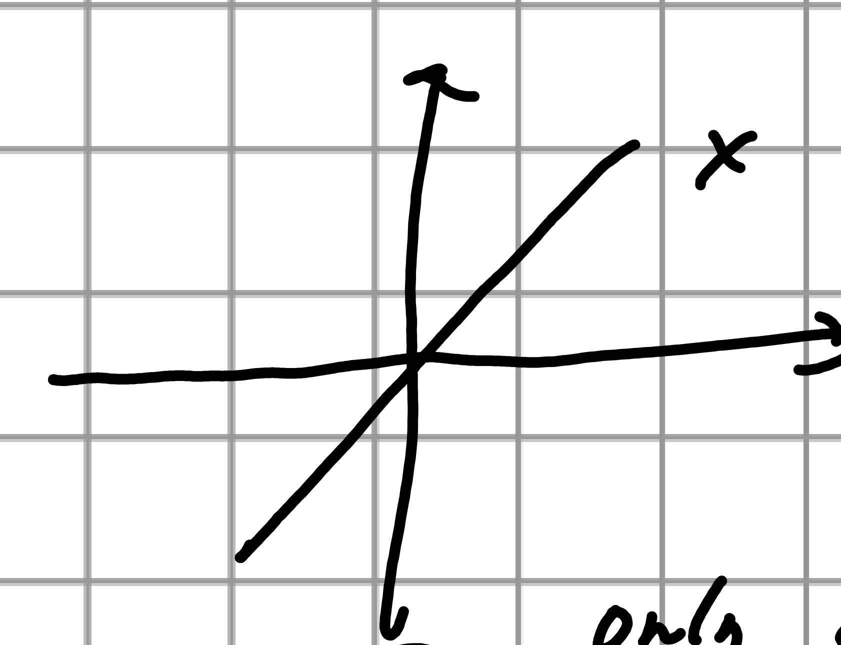

✗ only coercive.

Prop. $I_A : X \to \mathbb{R} \cup \{+\infty\}$ is weakly coercive

$\Leftrightarrow$ A is bdd.

Pf: ($\Leftarrow$) is trivial. For ($\Rightarrow$):

By contradiction. A is unbdd.

We can find $(x_n) \subset A$. $\|x_n\| \to +\infty$.

But $I_A(x_n) \equiv 0$. $\forall n$.

Thm. (Direct Method)

$F : X \to \mathbb{R} \cup \{+\infty\}$ proper, w.l.s.c. &

weakly coercive. Then $\exists x \in X$. s.t.

$$F(x) = \inf_{y \in X} F(y) < \infty.$$

Pf: Set (x_n) satisfies $F(x_n) \to \inf F$

Since $\inf F < \infty$ by proper.

So: $\|x_n\| < \infty$ from weakly coercive.

By reflexive of X. $\exists (n_k)$. s.t.

$x_{n_k} \rightharpoonup x$. So we have:

$F(x) \leq \varliminf F(x_{n_k}) = \inf F(y)$.

follows from w.l.s.c. of F.

Thm. (Obstacle problem)

For $\Omega \underset{\text{bdd}}{\subseteq} \mathbb{R}^n$. Lip domain. $f \in L^2(\Omega)$ and $\chi \in W^{1,2}(\Omega)$. St. $\chi \leq 0$. a.s. on $\partial\Omega$. Then. $\exists\, v \in W_0^{1,2}(\Omega)$. St. $v \geq \chi$ on Ω. uniquely minimizes:

$I(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 - fv\, dx + I_A(v)$

from $W_0^{1,2}(\Omega)$ to $\mathbb{R} \cup \{+\infty\}$. Where

$A := \{v \in W_0^{1,2}(\Omega) \mid v \geq \chi \text{ a.s. on } \Omega\}$.

Pf: Use Direct method on $I(v)$.

1) Properness: Note $v = 0 \vee \chi \in A$.

($Dv = D\chi$. a.s. on Ω. $\Rightarrow v \in W^{1,2}$)

2) W.l.s.c: Note $v \mapsto \frac{1}{2}\int |\nabla v|^2 - fv$ is conti. So l.s.c. And it's also convex. $\Rightarrow$ w.l.s.c.

For the part $I_A(v)$, we only need to check A is weakly closed ($\Leftarrow$ closed + convex. which's trivial ($v_n \to v$ in $W^{1,2} \Rightarrow \exists\, v_{n_k} \to v$ a.e.)

3) Weakly coercive :

$$I(v) \geq \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{2\varepsilon}\|f\|_{L^2}^2 - \varepsilon\|v\|_{L^2}^2$$

By Poincaré : $\|\nabla v\|_{L^2} \geq C\|v\|_{W_0^{1,2}} \to \infty$

also $\|\nabla v\|_{L^2} \geq \|v\|_{L^2}$. So $I(v) \to +\infty$