Intro. on ODEs

(1) <u>Definitions:</u>

<u>Def</u>: i) 1^{st} order scalar explicit ODE is

$$y'(t) = f(t, y(t))$$

where unknown $y_t : I \to D \subset \mathbb{R}^1$ is func. of time t. $f : I \times D \to \mathbb{R}^1$.

<u>Rmk</u>: "explicit" means it's not the form: $F(t, y'(t)) = 0$.

ii) System of 1^{st}-order ODEs is $\underline{y}'(t)$

$$:= \begin{pmatrix} y_1'(t) \\ \vdots \\ y_m'(t) \end{pmatrix} = \underline{f}(t, \underline{y}(t)) = \begin{pmatrix} f_1(t, y_1(t), \cdots y_m(t)) \\ \vdots \\ f_m(t, y_1(t), \cdots y_m(t)) \end{pmatrix}$$

where $\underline{y}_t : I \to D \subset \mathbb{R}^m$

and $\underline{f}(t) : I \times D \to \mathbb{R}^m$.

iii) An ODE is called autonomous if $f(t, x) = f(x)$ in i).

iv) 1^{st}-order scalar initial value problem

(IVP) is $y'(t) = f(t, y(t))$, $t \in [t_0, T]$, with $y'(t_0) = y_0$.

v) System of n^{th}-order ODE is:

$\underline{y}^{(n)}(t) = \underline{f}(t, \underline{y}(t), \underline{y}'(t), \cdots, \underline{y}^{(n-1)}(t))$.

$\underline{y}(t) \in \mathbb{R}^m$, $\underline{f} : I \times D \to \mathbb{R}^m$, $D \subset \mathbb{R}^{n \cdot m}$.

vi) IVP of n^{th}-order is v) on $[t_0, T]$ with $\underline{y}(t_0) = \underline{y}_0$, $\underline{y}'(t_0) = \underline{y}_1$, $\cdots$ $\underline{y}^{(n-1)}(t_0) = \underline{y}_{n-1}$.

__Rmk__: since $\underline{y}(t)$ is a m-dim vector so the initial value have dim $= n \cdot m$. And sometimes we also consider boundary value problem.

Note that most of numerical method is just designed for 1^{st}-order ODE. But when it comes to higher order, we can reduce it to 1^{st}-order:

Set $\underline{z}(t) := (\underline{z}_1(t) \cdots \underline{z}_n(t))^T := (\underline{y}(t) \cdots \underline{y}^{(n-1)}(t))$

from $[t_1, T]$ to $\mathbb{R}^{m \cdot n}$.

$\Rightarrow \underline{z}'(t) = (\underline{z}_2(t), \cdots \underline{z}_n(t), f(t, \underline{z}_1(t) \cdots \underline{z}_n(t)))^T$

And it's IVP with initial value:

$\underline{z}(t_0) = (y^0, y^1, \cdots y^{n-1})^T$.

(2) Well-posedness:

Thm. (Picard - Lindelöf)

$G \underset{\text{open}}{\subseteq} \mathbb{R}^1 \times \mathbb{R}^m$ domain. If $\underline{f}(t, y)$ is Lip in y with const. L in G. Set $(t_0, y_0) \in G$. $a, b > 0$. $R \stackrel{\Delta}{=} [t_0 - a, t_0 + a] \times \{\|y - y_0\| \le b\} \subset G$. $M \stackrel{\Delta}{=} \max_{R} \|\underline{f}(t, \underline{x})\|$. $\delta = a \wedge \frac{b}{M}$.

Then: IVP $y'(t) = \underline{f}(t, y(t))$. $y(t_0) = y_0$ has unique solution on $[t_0, t_0 + \delta]$.

e.g. $y'(t) = \sqrt{1 - y(t)^2}$ $\quad y(t_0) = y_0$. $t \in [0, 1]$.

For $(t_0, y_0) = (0, 0)$. We can apply Thm above on its nbd $\Rightarrow y(t) = \sin t$

But for $(t_0, y_0) = (0, -1)$. We note

$-x/\sqrt{1-x^2}$ isn't Lip around $x=-1$. So Thm above can't apply. Also it has at least two sol's $y(t)=-\cos(t)$ or $y(t)\equiv -1$.

Rmk: $y(t)$ should satisfy $y'(t)\geq 0$.

Thm. (stability)

$\underline{f}(t,\underline{x})$, $\underline{g}(t,\underline{x})$ are conti. on $I\times D$. If $\underline{f}(t,\underline{x})$ is Lip in $\underline{x}$ on D with const. L. Then, for IVPs:

$\underline{u}'(t) = \underline{f}(t,\underline{u}(t))$. $\underline{u}(t_0)=\underline{u}_0$. $t\in I$.

$\underline{v}'(t) = \underline{g}(t,\underline{v}(t))$. $\underline{v}(t_0)=\underline{v}_0$. $t\in I$.

there holds for any pair of sol's above: $\|\underline{u}(t)-\underline{v}(t)\|\leq e^{L(t-t_0)}[\|\underline{u}_0-\underline{v}_0\| + \int_{t_0}^{t}\varepsilon(s)ds]$. where $\varepsilon(t)\stackrel{\Delta}{=}\sup_{x\in D}\|\underline{f}-\underline{g}\|$.

Rmk: Let $\underline{g}=\underline{f}$. We can also obtain the well-posedness result above.

Pf: Set $\underline{e}(t) = \underline{u}(t)-\underline{v}(t)$

$$= \int_{t_0}^{t}(\underline{f}(s,\underline{u}(s)) - \underline{g}(s,\underline{v}(s)))ds + \underline{u}_0 - \underline{v}_0.$$

Pf: $\|\underline{e}(t)\| \le \int_{t_0}^{t} \|\underline{f}(s, \underline{u}(s)) - \underline{f}(s, \underline{v}(s))\|$

$+ \int_{t_0}^{t} \|\underline{f}(s, \underline{v}(s)) - \underline{g}(s, \underline{v}(s))\| ds + \|\underline{u}_0 - \underline{v}_0\|$

$\le \int_{t_0}^{t} L \|\underline{e}(s)\| + \varepsilon(s) ds + \|\underline{u}_0 - \underline{v}_0\|.$

Apply Gronwall's inequi. on it.

<u>Rmk</u>: We only require one of f, g

to be Lip-conti.