Extrapolation

(1) <u>Ideas</u>:

Note $\frac{f(x+h)-f(x)}{h} =: a_1(h) \xrightarrow{h\to 0} f'(x)$. For $f\in C^5$:

Apply Taylor expansion on $f(x+h)$ at x:

$a_1(h) = f'(x) + \frac{1}{2}hf''(x) + \frac{1}{6}h^2 f^{(3)}(x) + O(h^3)$.

i.e. $a_1(h)$ approxi. $f'(x)$ with first order

We can also use the central difference:

$a_2(h) := \frac{f(x+h)-f(x-h)}{2h}$

$\Rightarrow a_2(h) = f'(x) + \frac{1}{6}h^2 f'''(x) + O(h^4)$ by Taylor

So: $a_2(h)$ approxi. $f'(x)$ with second order

<u>Rmk</u>: In both case, we want to compute $a_i(h)$ when $h\to 0$ but can't set $h=0$

And $a(h)$ has expansion: $a(0) + a_m h^m + \cdots$

So the idea is:

i) For $h_0 > h_1 > \cdots > h_k > 0$. compute $a(h_i)$.

ii) Fit a polynomial $p(h)$ iii) Set $a(0) \approx p(0)$.

e.g. i) For fixed value H. And assume it has expansion:

$$a(H) = a(0) + c_m H^m + O(H^{m+1}).$$

$$a(\frac{H}{2}) = a(0) + c_m (\frac{H}{2})^m + O(H^{m+1}).$$

We interpolate them by linear poly $p(h) = b_0 + b_m h^m$ given by

$$b_0 = \frac{2^m a(\frac{H}{2}) - a(H)}{2^m - 1}, \quad b_m = 2^m \frac{a(H) - a(\frac{H}{2})}{H^m (2^m - 1)}$$

$\Rightarrow p(H) = a(H), \quad p(\frac{H}{2}) = a(\frac{H}{2})$

Then: set $p(0) = b_0 = a(0) + O(H^{m+1})$.

Rmk: We see that we increase order from $O(h^m)$ to $O(h^{m+1})$ by merely compute $a(h_i)$

ii) Note if we interpolate $a_1(h)$, $a_2(h)$ at $H, \frac{H}{2}$ by poly $\Rightarrow$ The order can be improved from 1 to 2 and 2 to 4.

Rmk: We notice in error expansion of a_2 only has even order power of h $\Rightarrow$ extrapolation is efficient.

Note above we only use $a(H)$, $a(\frac{H}{2})$ to fit a linear polynomial. More generally, we can fit a higher order poly. and delete high order error by using more values.

<u>e.g.</u> If we have $a(h) = a(0) + a_m h^m + a_{m+1} h^{m+1} + \cdots$

using $p(h) = b_0 + b_1 h^m + b_2 h^{m+1}$ from $a(H)$ and $a(\frac{H}{2})$, $a(\frac{H}{4})$. $\Rightarrow p(0)$ will approx. of order $m+2$

(2) <u>Applications on ODEs:</u>

We can try to apply the idea on local error i.e. solve IVP with step length H and solve it again with $\frac{H}{2}$ to get y_n^H, $y_n^{\frac{H}{2}}$.

Note for $f \in C^{p+1}$. Apply one-step method of order m with equidistant step length h $\Rightarrow$ we have: $y_n = y(t_n) + h^m e_m(t_n) + \cdots h^N e_N(t_n) + O(h^{N+1})$

<u>Rmk</u>: Note equidistant time stepping means it's not allowed to use adaptive time stepping

$\Rightarrow$ We use extrapolation locally in each step e.g. From step k to $k+1$:

$y_k \xrightarrow{h} y_{k+1}^{h}$. $y_k \xrightarrow{\frac{h}{2}} \tilde{y}^{\frac{h}{2}} \xrightarrow{\frac{h}{2}} y_{k+1}^{\frac{h}{2}}$. Then we estimate

error by $EST_{k+1} = |y_{k+1}^{\frac{h}{2}} - y_{k+1}^{h}| / 1-2^{-m}$

Also, we interpolate by y_{k+1}^{h}. $y_{k+1}^{\frac{h}{2}}$ to get

$p(0) = (2^m y_{k+1}^{\frac{h}{2}} - y_{k+1}^{h}) / 2^m - 1 \approx y_{k+1}^{h \to 0}$ (i.e. exact sol.)

Then we estimate the local error as

$|y_{k+1}^{h} - p(0)| = |\frac{2^m}{2^m-1} \cdot (y_{k+1}^{h} - y_{k+1}^{\frac{h}{2}})|$.

One method suitable to use extrapolation if

i) It's cheap. (litte eval. of f).

ii) Expansion only contains even power of h

can be efficient.

e.g. Most commonly used method on extrapolation is explicit midpoint rule:

$y_n = y_{n-2} + 2hf(t_{n-1}, y_{n-1})$ started with

explicit Euler.

Thm. (Gragg)

For $f \in C^{2m+2}$. Consider mid-pt rule started with explicit Euler. Then: $y_n = y(t_n) + \sum_{k=1}^{m} h^{2k} (a_k(t_n) + (-1)^n b_k(t_n)) + O(h^{2m+2})$

(i.e. it has expansion only in even power)

Rmk: i) Note that there are some techniques to deal with oscillatory of $b_k(t_n)$.

In each step: e.g. $\{n_i\} = \{2, 4, 6, 8, 12, 16\}$.

For step $t_k \xrightarrow{H} t_{k+1}$:

Compute with n_i steps of length $\frac{H}{n_i}$

values $\{y_{k+1}^{(i)}\}_i$;

$\Rightarrow$ use the gained values to apply extrapolation

Then: we find the local truncated error has increased from order 2 to order 12 for f smooth enough.

ii) The Gragg's extrapolation method can be seen as one-step method with step length H. Since it starts from one-step expl. Euler