Galerkin's & Multidim

(1) Galerkin's Scheme & Basis:

Def: $(V, \|\cdot\|)$ Banach. $(V_n) \subset V$ seq of finite dim subspace is called Galerkin's scheme if $d(v, V_n) := \inf_{u \in V_n} \|v - u\| \xrightarrow{n\to\infty} 0$. $\forall v \in V$.

Rmk: $\overline{\bigcup V_n} = V$. i.e. $\bigcup V_n$ is dense.

$(\phi_n) \subset V$ is a Galerkin's basis if $V_n = \mathrm{span}\{\phi_k\}_1^n$ and $\{\phi_i\}_1^m$ are l.i. $\forall m$.

Rmk: (ϕ_n) or (V_n) may not exist.

Lem. If V is sep. Then: Galerkin's basis exists

Pf: $\exists (V_n)$ dense subset of U.

And we extract its l.i. subset by choose $\phi_j = U_{kj}$. s.t. $V_{kj} \notin \mathrm{span}\{\phi_p\}_1^{j-1}$

e.g. $\phi_n(x) = x^{n-1}$ is Galerkin's basis for $C[a,b]$ by Weierstrass-Stone Thm.

Rmk: L^∞ isn't separable so there is no

Galerkin's scheme. By contradict:

If (ϕ_n) is Galerkin's basis. Set $\widetilde{W}_n = \mathrm{span}_{\mathbb{R}}\{\phi_1, \cdots, \phi_n\}$. $\Rightarrow \cup \widetilde{W}_n \overset{\text{dense}}{\subset} \cup W_n = L_\infty$.

e.g. $A: V \to V^*$ strongly positive BLO. For $f \in V^*$.

We want to solve $Au = f$ in V.

Set $a(u, v) = \langle Au, v \rangle = \langle f, v \rangle$.

$\Rightarrow$ Consider discretized problem: $u_n, v_n \in V_n$.

$a(u_n, v_n) = \langle f, v_n \rangle$ in V_n. Galerkin's scheme.

Set $a_n: V_n \times V_n \to \mathbb{R}$. $a_n(u_n, v_n) = a(u_n, v_n)$

for $u_n, v_n \in V_n$. is strongly positive BLO.

Apply Lax-Milgram: $\exists u_n$. solve the discretized problem. We collect (u_n). hope $u_n \to u$ in V.

Lem of Céa: u is sol. for $a(u, v) = \langle f, v \rangle$.

Then: $\|u_n - u\| \leq \beta \mu^{-1} d(u, V_n)$. β, μ are const. from BLO & strongly positive.

Pf: $a(u_n, v_n) = \langle f, v_n \rangle = a(u, v_n)$ by def.

$\Rightarrow a(u - u_n, v_n) = 0$. $\forall v_n \in V_n$. (a-ortho.)

So: $a(u-u_n) = a(u-u_n, u-v_n)$. $\forall v_n \in V_n$.

$LHS \geq \mu \|u-u_n\|^2$. $RHS \leq \beta \|u-u_n\| \|u-v_n\|$.

Rmk: LHS is discretization error and

RHS is approxi. error.

So for Galerkin's scheme, $u_n \xrightarrow{\|\cdot\|} u$.

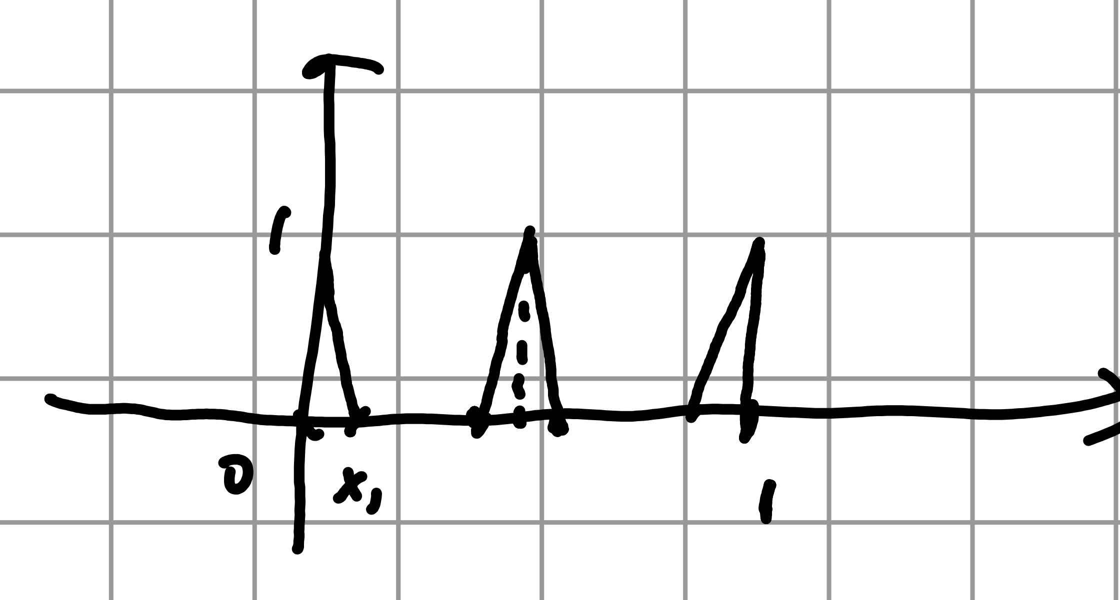
(2) 1-dim Finite Element Method:

Consider $-u'' = f$ on $(0,1)$. $u(0) = u(1) = 0$.

Set $x_k = k/n \stackrel{\Delta}{=} kh$. $0 \leq k \leq n$. $\varphi_0(x) = \begin{cases} h^{-1}(x_1 - x). & x_1 \geq x \geq 0 \\ 0 & . \text{ else} \end{cases}$

$$\varphi_i(x) = \begin{cases} h^{-1}(x - x_{i-1}). & x_{i-1} \leq x \leq x_i \\ h^{-1}(x_{i+1} - x). & x_i \leq x \leq x_{i+1} \\ 0 & . \text{ else} \end{cases} \quad \text{for } 1 \leq i \leq n-1.$$

$$\varphi_n(x) = \begin{cases} h^{-1}(x - x_{n-1}). & x_{n-1} \leq x \leq x_n \\ 0. & \text{else} \end{cases}$$



$\Rightarrow \varphi_i \in H^1(0,1)$. $\forall i$. and $\varphi_k \in H_0^1(0,1)$. $1 \leq k \leq n-1$

Set $V_n = \text{span}\{\varphi_i\}_1^{n-1} \subset H_0^1(0,1)$.

Rmk: $V_n \not\subset V_{n+1}$. $\forall n$. (Consider discontinuity)

Actually we can prove (V_n) is Galerkin's scheme.

To solve $a(u_n, v_n) = \langle f, v_n \rangle$, $\forall v_n \in V_n$. $\Leftrightarrow$

$$a(u_n, \varphi_k) = \langle f, \varphi_k \rangle, \ \forall 1 \le k \le n-1.$$

Since $u_n = \sum_{1}^{n-1} u_n^i \varphi_i \in V_n$, $u_n^i \in \mathbb{R}$. So it's equi.:

$$\sum_{i=1}^{n-1} u_n^i \, a(\varphi_i, \varphi_k) = \langle f, \varphi_k \rangle, \ 1 \le k \le n-1.$$

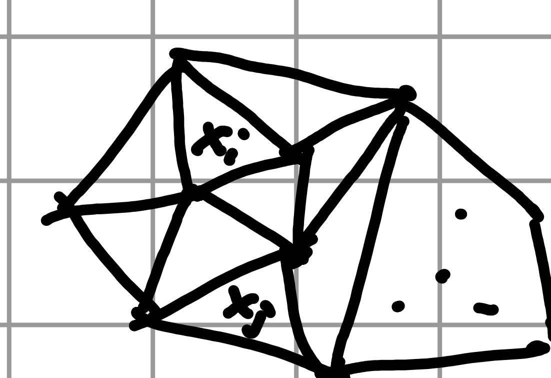
i.e. solve $A_n \bar{u}_n = f_n = (\langle f, \varphi_k \rangle)_1^{n-1}$ where $A_n = (a(\varphi_k, \varphi_j))_{k,j}$. $\bar{u}_n = (u_n^i)_1^{n-1}$.

Note $a(\varphi_k, \varphi_j) = \int_0^1 \varphi_k' \varphi_j' \, dx = \frac{2}{h} I_{\{i=j\}} - \frac{1}{h} I_{\{j=i-1 \text{ or } i+1\}}$

$a(w, \varphi_i) = \int_0^1 w'(x) \varphi_i'(x) = \frac{1}{h}(2w(x_i) - w(x_{i-1}) - w(x_{i+1}))$

$$\Rightarrow A_n = h^{-1} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \Rightarrow \text{We can solve } \bar{u}_n.$$

Rmk: When $d > 1$. We define φ_k in triangle func.:



$\Rightarrow u_n^i = u(x_i)$. i.e. We have:

$$a(u - I_n u, \varphi_i) \overset{\text{ortho}}{=} a(u_n - I_n u, \varphi_i) = 0. \ \forall i.$$

where $I_n u = \sum_{1}^{n-1} u(x_k) \varphi_k$. $I_n : V \to V_n$ is a interpolation operator. where $V = H_0^1(a,b)$

With $d(u, V_n) = \inf_{V_n} \|u - v\| \le \|u - I_n u\| \xrightarrow{\text{prop.}} 0$.

Rmk: If consider Galerkin's scheme in $H^1(0,1)$.

$\Rightarrow$ We will use φ_0, φ_n as well.

<u>prop</u>: $|u_n - u|_{1,2} \xrightarrow{n\to\infty} 0$. $\forall u \in V = H_0^1(0,1)$.

<u>pf</u>: 1) prove I_n is BLO:

$$|I_n u|_{1,2}^2 = \sum_1^n \int_{x_{i-1}}^{x_i} |(I_n u)'|^2 dx.$$

$$= \sum_1^n \int_{x_{i-1}}^{x_i} h^{-2} |u(x_i) - u(x_{i-1})|^2 dx$$

$$= \sum_1^n \int_{x_{i-1}}^{x_i} h^{-2} \left(\int_{x_{i-1}}^{x_i} u'(y) dy\right)^2 dx$$

$$\overset{\text{Hölder}}{\leq} \sum_1^n h \cdot h^{-2} \int_{x_{i-1}}^{x_i} |u'|^2 dy \cdot h = |u|_{1,2}.$$

2) $\exists \tilde{u} \in C_c^\infty$. s.t. $|u - \tilde{u}|_{1,2} < \varepsilon$.

$$\Rightarrow |u - I_n u|_{1,2} \leq |u - \tilde{u}|_{1,2} + |\tilde{u} - I_n \tilde{u}|_{1,2} + |I_n(u - \tilde{u})|_{1,2}.$$

$$\overset{1)}{\leq} C\varepsilon + |\tilde{u} - I_n \tilde{u}|_{1,2}$$

$$|\tilde{u} - I_n \tilde{u}|_{1,2}^2 = \sum_1^{n+1} \int_{x_{i-1}}^{x_i} \left(\tilde{u}'(x) - \int_{x_{i-1}}^{x_i} h^{-1} \tilde{u}'(t) dt\right)^2 dx$$

$$= \sum_1^{n+1} \int_{x_{i-1}}^{x_i} h^{-2} \left(\int_{x_{i-1}}^{x_i} \tilde{u}'(x) - \tilde{u}'(t) dt\right)^2 dx.$$

$$= \sum_1^{n-1} \int_{x_{i-1}}^{x_i} h^{-2} \left(\int_{x_{i-1}}^{x_i} \int_t^x \tilde{u}''(p) dp\, dt\right)^2 dx.$$

$$\overset{\text{Hölder}}{\leq} \sum_1^{n-1} \int_{x_{i-1}}^{x_i} h^{-2} \cdot h \int_{x_{i-1}}^{x_i} \underbrace{(x-t)}_{\leq h} \int_t^x \tilde{u}''(p)^2 dp\, dt\, dx.$$

$$\leq \sum_1^{n-1} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} |\tilde{u}''(p)|^2 dp\, dt\, dx$$

$$\leq h^2 \int_0^1 |\tilde{u}''(p)|^2 dp = h^2 \|\tilde{u}''\|_{0,2}^2.$$

Rmk: Note we obtain $|\tilde{u}-I_h\tilde{u}|_{1,2} \le h\|\tilde{u}''\|_{0,2}$

finally. But we still need to add

$O(\varepsilon)$ on $|u-I_h u|_{1,2} \Rightarrow$ No decay speed.

Actually, if consider $H_0^2(0,1)$-func.

$\Rightarrow$ it really has linear decay.

(3) Multidimension case:

Fix $\mathbb{R}^d$. $\alpha=(\alpha_1,\cdots\alpha_d)\in\{0,1,\cdots\}^d$ is multindex.

Set $|\alpha|=\sum_1^d \alpha_i$. $D^\alpha=\prod_1^d \partial^{\alpha_i}/\partial x_i^{\alpha_i}$. $\alpha!=\prod_1^d \alpha_i!$.

For $h\in\mathbb{R}^d$. $h^\alpha \triangleq \prod_1^d h_i^{\alpha_i}$.

Def: $\Omega\subseteq\mathbb{R}^d$ bdd open domain. $\alpha\in\mathbb{N}^d$. $u,v\in$

$L^1_{loc}(\Omega)$. st. $\int_\Omega u D^\alpha\varphi\, dx=(-1)^{|\alpha|}\int_\Omega v\varphi\, dx$. $\forall\varphi\in C_c^\infty$

if v is α^{th}-weak deri. of u.

Lem. (Funda. Thm of Cal. of Var.)

$\int_\Omega u\varphi\, dx=0$. $\forall\varphi\in C_c^\infty(\Omega)$ for $u\in L^1_{loc}(\Omega)$

$\Rightarrow u=0$. a.e on Ω.

Pf: Still by regularization: $J(x)=I_{\{|x|\le1\}}e^{\frac{1}{1-|x|^2}}$

$J_\varepsilon(x)=C_\varepsilon J(\frac{x}{\varepsilon})$. st. $\int_{\mathbb{R}^d}J_\varepsilon(x)=1$.

Let $u_\varepsilon=J_\varepsilon * u \xrightarrow{L^1} u$.

Def: $W^{k,p}(\Omega) := \{$weak fun. $D^\alpha u \in L^p(\Omega)$. for $\forall \alpha \in \mathbb{N}^d$. s.t. $|\alpha| \le k\}$. $1 \le p \le \infty$. with norm

$\|u\|_{k,p} := \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_{0,p}^p \right)^{\frac{1}{p}}$. $1 \le p < \infty$.

$\|u\|_{k,\infty} := \sup_{|\alpha| \le k} |D^\alpha u|$

Seminorm $|u|_{k,p} := \left(\sum_{|\alpha| = k} \|D^\alpha u\|_{0,p}^p \right)^{1/p}$.

$|u|_{k,\infty} := \sup_{|\alpha| = k} |D^\alpha u|$.

$W_0^{k,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{k,p}} \subsetneq W^{k,p}(\Omega)$. cls.

Rmk: But $W^{k,p} \not\supseteq C^\infty(\Omega)$. Deri. may explode when $x \to \partial\Omega$.

Thm. (Meyers - Serrin)

For bdd domain $\Omega \subset \mathbb{R}^n$ and $\forall 1 \le p < \infty$.

$W^{k,p}(\Omega) = \overline{W^{k,p}(\Omega) \cap C^\infty(\Omega)}^{\|\cdot\|_{k,p}} =: H^{m,p}$

Rmk: For $1 \le p < \infty$. $\Omega = \mathbb{R}^n$. $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$. So:

$W_0^{m,p}(\mathbb{R}^n) = H^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$.

Thm. $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ is Banach. Besides, it's separa. if $1 \le p < \infty$ and reflexive if $1 < p < \infty$.

Rmk: For $1<p<\infty$, $H^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ are sep. and reflexive.

Thm. $H^k(\Omega) := W^{k,2}(\Omega)$ is Hilbert space with inner product. $((u,v))_{H^k} := \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)_{L^2}$

Rmk: Similarly set $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

Thm. (Pioncáre)

For $k\in\mathbb{N}$, $1\le p<\infty$. There exists $C>0$. st.

$\|D^\alpha u\|_{0,p} \le C|u|_{k,p}$ $\forall|\alpha|<k$. $\forall u\in W_0^{k,p}(\Omega)$.

Def: $H^{-1}(\Omega) = (H_0^1(\Omega))^*$.

Def: A domain $\Omega \overset{bdd}{\subseteq} \mathbb{R}^n$ is Lip-conti. if $\forall x_0 \in\partial\Omega$. $\exists$ Lip conti. func. $g:\mathbb{R}^{n-1}\to\mathbb{R}^1$ and $r>0$

$\Omega\cap B(x_0,r) = \{x\in B(x_0,r) \mid g(x_1,\cdots x_{n-1}) < x_n\}$

up to rotation.

Rmk: It implies $\partial\Omega\cap B(x_0,r) = \{x\in B(x_0,r) \mid x_n = g(x_1,\cdots x_{n-1})\}$.

Thm. For Ω Lip. domain $\Rightarrow$ $C^\infty(\bar\Omega)$ is dense in $W^{k,p}(\Omega)$. $\forall 1\le p<\infty$

Rmk: $C^\infty(\Omega)$ won't work. (Not subset.)

Embedding:

$W^{k,q} \hookrightarrow W^{k,p}$. $W^{k,p} \hookrightarrow W^{l,p}$. $\forall q \geq p$. $k \geq l$. But we can also trading differentiability and integrability: Sobolev embedding.

Thm. $\Omega \subset \mathbb{R}^d$. Lip domain. $k \in \mathbb{N}$. $p \in [1, \infty]$.

i) If $kp < d$. Then: $W^{k,p} \hookrightarrow W^{l,q}$ for $\forall$ $k \geq l$ and $q^{-1} \geq p^{-1} - \frac{k-l}{d}$.

ii) If $kp = d$. $W^{k,p} \hookrightarrow L^q$. $\forall q < \infty$. (But not in L^∞ generally).

iii) If $kp > d$. Then: $W^{k,p}(\Omega) \hookrightarrow C^{k - \lfloor \frac{d}{p} \rfloor - 1, \alpha}(\bar{\Omega})$

where $\alpha \in \begin{cases} (0,1). & \text{if } d/p \in \mathbb{N}. \\ (0, \lfloor \frac{d}{p} \rfloor + 1 - \frac{d}{p}). & \text{else.} \end{cases}$

iv) If $kp < d$. Then: $W^{k,p} \overset{cpt}{\hookrightarrow} W^{l,q}$ for $q^{-1} > p^{-1} - \frac{k-l}{d}$. $\forall k \geq l$.

(The embedding in iii) can also be cpt)

Rmk: It still holds when replacing $W^{k,p}$ by $W_0^{k,p}$ and using general domain.

e.g. For $\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases}$ on Lip domain $\Omega \subset \mathbb{R}^n$.

Lem. (Partial integration)

On Lip. domain $\Omega \subset \mathbb{R}^n$. For $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, $\varphi \in C^1(\mathbb{R}^n; \mathbb{R}^1)$. Then:

$\int_\Omega \mathrm{div}(F)\, \varphi \, dx = \int_{\partial\Omega} \varphi F \cdot \nu \, ds - \int_\Omega F \cdot \nabla\varphi \, dx$. ν is outer normal vector of $\partial\Omega$.

So it's reduced to find $u \in H_0^1(\Omega)$. s.t. u satisfies the weak formulation:

$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$ for $\forall v \in H_0^1(\Omega)$.

<u>Trace operator:</u>

For $\Omega \subset \mathbb{R}^n$. Lip domain. Note $C^\infty(\bar\Omega)$ is dense in $W^{k,p}(\Omega)$. Since $u \in C^\infty(\bar\Omega)$. $u|_{\partial\Omega} \in C(\partial\Omega) \cap L^1(\partial\Omega)$ is well-defined. We can show:

$\|u|_{\partial\Omega}\|_{L^1(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$. $\forall u \in C^\infty(\bar\Omega)$.

So $Tr: C^\infty(\bar\Omega) \subset W^{1,p}(\Omega) \to L^1(\partial\Omega)$ is BLO.

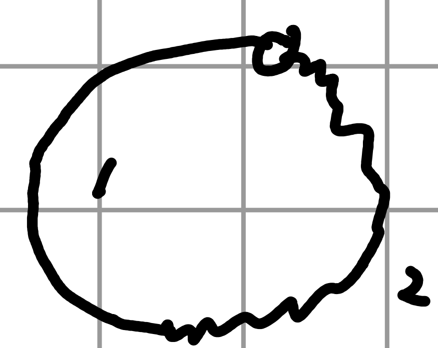
$$u \mapsto u|_{\partial\Omega}$$

which has unique extension on $W^{1,p}(\Omega)$.

$\Rightarrow$ Boundary condition $u|_{\partial\Omega}=0$ for $u \in W^{1,p}(\Omega)$

is defined by $Tr(u)=0$.

Rmk: Tr isn't injective or surjective. e.g.



It will steep increase on pt "0" so not in $R(Tr)$.

But $Tr(W^{1,p}(\Omega)) = W^{1-p^{-1},p}(\partial\Omega)$

Thm. $W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \mid Tr\,u = 0\}$.