Weak Deri. & Sobolev

(1) Weak Deri.:

Def: $u, v \in L^1_{loc}(a,b)$. s.t. $\int v \varphi' dx = -\int u \varphi dx$ for $\forall \varphi \in C_c^\infty(a,b)$. Then: we say u is weak deri. of v.

Rmk: i) u is unique. And if $v \in C^1$ or AC. then: $u = v'$

ii) Weak deri. $(\cdot)'$ is linear.

iii) Weak deri. is said to def on interval/open set. rather point.

eg. i) $V: (-1,1) \to \mathbb{R}^1$. $v(x) = |x| \notin C^1$. But it has weak deri. $u = I_{\{x>0\}} - I_{\{x<0\}}$. which can be easily checked.

ii) $V: (-1,1) \to \mathbb{R}^1$. $V(x) = I_{\{x>0\}} - I_{\{x\le 0\}}$. isn't weakly differentiable.

Set $\varphi(x) = \begin{cases} \exp(-1/1-x^2). & x \in (-1,1) \\ 0 & . \text{ else}. \end{cases}$

$\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$. if $\exists u \in L^1_{loc}(-1,1)$

St. $-\int_{-1}^{1} u(x)\,\varphi_\varepsilon(x) = \int_{-1}^{1} v(x)\,\varphi_\varepsilon'(x)$

$RHS = \int_0^1 \varphi_\varepsilon'\,dx = e^{-1}$.

But $|LHS| = |\int_{-1}^{1} \chi_{(-\varepsilon,\varepsilon)}\, u\, \varphi_\varepsilon|$

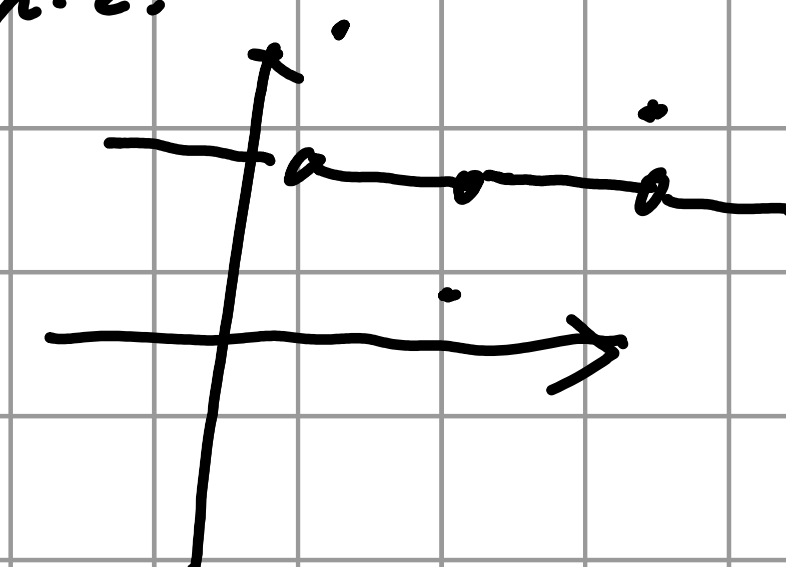
$\le \frac{1}{e} \int_{-1}^{1} \chi_{(-\varepsilon,\varepsilon)} |u| \xrightarrow{\varepsilon\to 0} 0$. contradict!

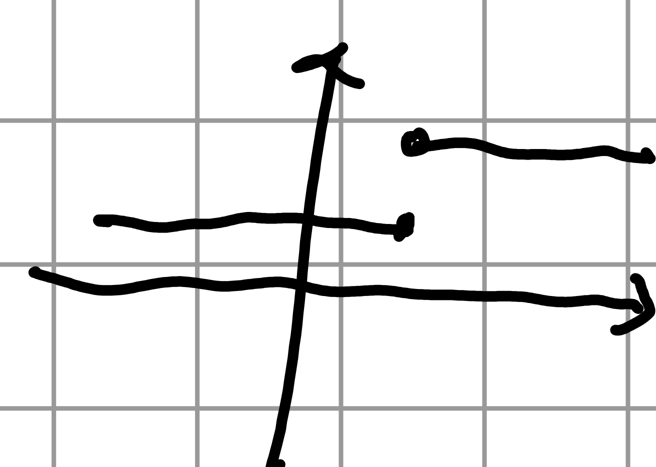
<u>Lem.</u> $u \in L^1_{loc}(a,b)$. $\int_a^b u\varphi = 0$. $\forall \varphi \in C_c^\infty(a,b)$

$\Rightarrow u = 0$ a.e.

<u>Cor.</u> $u \in L^1_{loc}(a,b)$. $\int_a^b u\varphi' = 0$. $\forall \varphi \in C_c^\infty(a,b)$

u = const func. a.e.

<u>RMK:</u> It means  rather

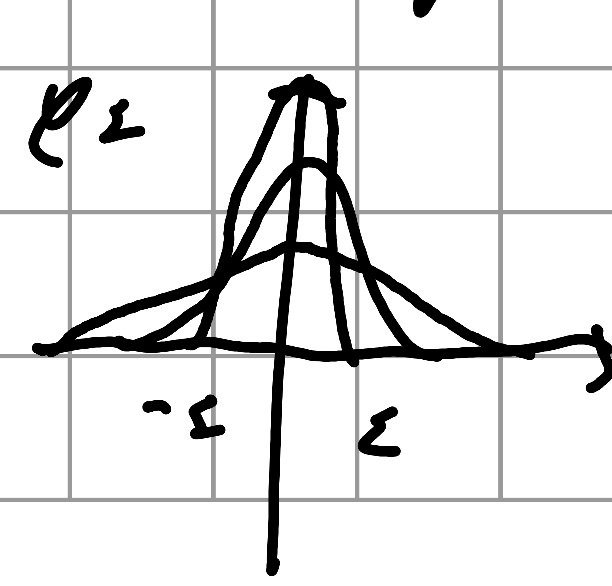
 (not a.e. const.!)

<u>Pf:</u> We want to find $(\varphi_n) \to \mathrm{sgn}(u)$

Then $\int_a^b u\varphi = \int_a^b |u| = 0$.

Choose $\varphi_\varepsilon = C_\varepsilon \psi_\varepsilon$. ψ_ε is func.

in e.g. ii). And $C_\varepsilon^{-1} = \int_a^b \psi_\varepsilon\,dx$.



Set $V_\varepsilon = \varphi_\varepsilon * U$. where

$V \in L^p$ for $1 \le p < \infty$. Then:

i) $V_\varepsilon \in C^\infty$ ii) $\|V_\varepsilon\|_{0,p} \le \|V\|_{0,p}$

iii) $\mathrm{supp}(V) \underset{cpt}{\subset} (a,b) \Rightarrow V_\varepsilon \in C_c^\infty(a,b)$ for ε small enough.

iv) $\|V_\varepsilon - V\|_{0,p} \to 0$ & $V_\varepsilon \to V$ a.e.

v) $V_\varepsilon \xrightarrow{u.c.} V$ if V is conti.

Pf: i) By DCT.

ii) $\int |\int \varphi_\varepsilon(y) V(x-y) dy|^p$

$= \int |\int \varphi_\varepsilon^{\frac{1}{q}}(y) \varphi_\varepsilon^{\frac{1}{p}} V(x-y) dy|^p$

$\underset{\int \varphi_\varepsilon = 1}{\overset{\text{Hölder}}{\le}} \int \varphi_\varepsilon(y) (\int |V(x-y)|^p dx) dy.$

$= \|V\|_{0,p}^p \int \varphi_\varepsilon = \|V\|_{0,p}^p$

iii) $V_\varepsilon(x) = \int_{x-\varepsilon}^{x+\varepsilon} \varphi_\varepsilon(x-y) V(y) dy.$

So: $V_\varepsilon = 0$ if $x \notin \mathrm{supp}(V)$ and $\mathrm{dist}(x, \mathrm{supp}(V)) > \varepsilon$. So:

$\mathrm{supp}(V_\varepsilon) \subset\subset \mathrm{supp}(V) + [-\varepsilon, \varepsilon]$

iv) $\|V_\varepsilon - V\|_{0,p}^p = \int |\int \varphi_\varepsilon(y)(u(x-y) - u(x))|^p$

$$\overset{\text{Hölder}}{\leq} \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(y) \int |V(x-y) - V(x)|^p \, dx \, dy$$

$$\leq \sup_{|y|<\varepsilon} \int |V(x-y) - V(x)|^p \, dx \xrightarrow{\varepsilon \to 0} 0$$

(Continuity of L^p-integral. by $C_c \underset{\text{dense}}{\subset} L^p$ via truncation)

$$|V_\varepsilon(x) - V(x)| \underset{C_\varepsilon = C\varepsilon^{-1}}{\leq} 2Ce^{-1} \cdot \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |V(x-y) - V(x)|$$

$\to 0$ by Lebesgue Diff. Thm.

v) By uniformly conti. on cpt set $\tilde{K} = \overline{K + [-\varepsilon, \varepsilon]}$.

Rmk: L^∞ isn't separable. which is also from the fact it's not well-approxi

Return to the proof: Set $[c,d] \subset (a,b)$.

And let $W(x) = \operatorname{sgn}(u(x)) \chi_{[c,d]}(x)$.

$J_\varepsilon(x) := W * \varphi_\varepsilon(x) \to W$. a.e.

$$|u(x) J_\varepsilon(x)| \leq |u(x) \chi_{[c-\varepsilon_0, d+\varepsilon_0]}| \overset{\text{Fix } \varepsilon_0 > \varepsilon}{\in} L^1.$$

Apply DCT on $\int_a^b u(x) J_\varepsilon(x) \to \int_c^d |u| = 0$

(2) Sobolev space:

Def: For $1 \le p \le \infty$. $W^{1,p}(a,b) \triangleq \{u \in L^p(a,b) \mid$ weak deri. u' exists. and $u' \in L^p(a,b)\}$

Rmk: i) $C^\infty(a,b) \not\subset W^{1,p}(a,b)$ necessarily.

ii) $W^{1,p}(a,b)$ is n.v.s. equipped with

$$\|u\|_{1,p} := \begin{cases} (\|u\|_{0,p}^p + \|u'\|_{0,p}^p)^{\frac{1}{p}}, & 1 \le p < \infty \\ \|u\|_{0,\infty} \vee \|u'\|_{0,\infty}. & p = \infty. \end{cases}$$

Rmk: $\|u'\|_{0,p}$ is seminorm in $W^{1,p}$.

Denote: $H^1(a,b) := W^{1,2}(a,b)$ equipped with inner product $((u,v))_{1,2} := \int_a^b uv + u'v' \, dx$.

Lem.

$$W^{1,p}(a,b) \text{ is } \begin{cases} \text{Banach for } 1 \le p \le \infty \\ \text{Separable for } 1 \le p < \infty \\ \text{reflexive for } 1 < p < \infty. \end{cases}$$

Pf: i) Complete: (u_n) is $W^{1,p}$ - Cauchy

$\Rightarrow (u_n), (Du_n)$ are L^p - Cauchy

$\exists\, u, v \in L^p$. $u_n \xrightarrow{L^p} u$. & $Du_n \xrightarrow{L^p} v$.

$\Rightarrow Du_n$ and u are L_p-u.i.

Also, $\forall \phi \in C_c^\infty$, $|\phi Du_n| \leq M|Du_n|$

$\Rightarrow \phi Du_n$ is L^p-u.i.

$\int_0$: $\int u D\phi = \lim_n \int u_n D\phi$

$= \int v\phi. \Rightarrow u' = v.$

2) Separability: $T : W^{1,p}(a,b) \to L^p(a,b)^{\oplus 2}$

define by $Tu = (u, Du)$

Note T is Linear isometry with

$W^{1,p}(a,b)$ is complete

$\Rightarrow T(W^{1,p}(a,b)) \underset{cls}{\subseteq} L^p \times L^p$

<u>Def</u>: X, Y are n.v.s. Y is embedded in X if

$\exists$ linear injection $i : Y \to X$.

i is conti. embedding if it is conti.

And we write $i : Y \hookrightarrow X$.

<u>RMK</u>: If $Y \subset X$. $\Rightarrow i = id : Y \hookrightarrow X$ is conti.

embedding if $\|i(y)\|_Y = \|y\|_X \leq C\|y\|_Y$.

<u>e.g.</u> $W^{1,p} \hookrightarrow L^p$. (even cpt embed)

Def: $Y \overset{cpt}{\underset{i}{\hookrightarrow}} X$ if the embedding is cpt.

Rmk: It mean bdd seq $\{f_n\} \subset Y$ will be seq-cpt w.r.t. the weaker norm $\|\cdot\|_X$

$Y \overset{dense}{\underset{i}{\hookrightarrow}} X$ if $i(Y) \subset X$ is dense subset.

Thm. For $u \in W^{1,1}(a,b)$. Then: $u = \tilde{u}$. a.e. where $\tilde{u} \in AC[a,b]$.

Besides, $\exists\, c > 0$. st. $\|\tilde{u}\|_{C[a,b]} \le c\|u\|_{W^{1,1}}$. Where c is indept of choice of $u \in W^{1,1}(a,b)$.

Rmk: i) $AC[a,b]$ can be defined on boundary pt

ii) AC can be interpreted:

For any finite pair $\{(t_n^1, t_n^2)\}$. st. its total measure $\sum|t_n^2 - t_n^1| < \delta$.

$\Rightarrow \sum|u(t_n^2) - u(t_n^1)| < \varepsilon$. i.e. uniform conti. on any finite pairs.

iii) u equal to $\tilde{u} \in C(a,b)$. a.e. $\neq$ u is a.e-conti. e.g.

a) $u = I_{\{x \geq 0\}}$ is a.e.-conti. but no $\tilde{u} \in C(\mathbb{R}^1)$. s.t. $\tilde{u} = u$. a.e.

b) $u = I_{\mathbb{Q}} \overset{a.e.}{=} 0$. But it is not a.e. - Conti.

Cor. $\|u\|_a^2 := |u(a)|^2 + |u|_{1,2}^2$ defines a equi. norm on $(H^1(a,b), \|\cdot\|_{1,2})$.

Pf: $\|\cdot\|_a$ is norm by FTCV.

$$\text{And } \|u\|_a^2 \leq \|u\|_{C[a,b]}^2 + \|u\|_{1,2}^2$$
$$\leq (c^2+1)\|u\|_{1,2}^2$$

$$\|u\|_{0,2}^2 = \int_a^b |u(a) + \int_a^x u'(t)dt|^2 dt$$
$$\leq C(a,b)\|u\|_a^2.$$

Pf: Set $v = \int_a^x u'(t)dt. \in AC[a,b]$.

So: it's a.e. - diff. by Lebesgue.

Note $\int u\varphi' \overset{IbP}{=} -\int u'\varphi = -\int v'\varphi.$
$$\overset{IbP}{=} \int v\varphi'. \quad \forall \varphi \in C_c^\infty.$$

$\Rightarrow u = v + \text{const}.$ a.e.

And $u(x) = u(x_0) + \int_{x_0}^x u'(t)dt$ since

$u \in AC[a,b]$. where x_0 satisfies:

$u(x_0) = (b-a)^{-1} \int_a^b u(t)\,dt$ (IVT).

$\Rightarrow |u(x)| \le ((b-a)^{-1} \vee 1) \int_a^b |u| + |u'|\,dt$.

$\le C \|u\|_{W^{1,1}}$. $\forall x \in (a,b)$.

Cor. $W^{1,1}(a,b) \hookrightarrow C([a,b]) \hookrightarrow L^{\tilde{p}}(a,b)$

$\bigcup$

$W^{1,p}(a,b)$ for $\forall \tilde{p}, p \in [1,\infty]$.

Thm. (Rellich)

$W^{1,p}(a,b) \overset{cpt}{\hookrightarrow} C[a,b]$. $\forall p > 1$.

Pf: Let $A \subset W^{1,p}(a,b)$ bdd by M.

To apply Ascoli Lem. :

$|u|_\infty \le C\|u\|_{W^{1,1}} \le C\|u\|_{W^{1,p}} \le CM$.

$|u(x) - u(y)| \le |\int_y^x u'(t)\,dt| \le \|u'\|_{0,p} |x-y|^{1/2}$

$\le \|u\|_{W^{1,p}} |x-y|^{1/2} \le M |x-y|^{1/2}$.

Rmk: $W^{1,1}(a,b)$ isn't cptly embedded into $C[a,b]$

Cor. For $0 < \tau < 1/2$. $H^1(a,b) \overset{cpt}{\hookrightarrow} C^{0,\tau}[a,b]$.

Pf: Lem. $C^{0,\beta}[a,b] \overset{cpt}{\hookrightarrow} C^{0,\tau}[a,b]$. $\forall 0 < \tau < \beta < 1$

Pf: By Ascoli. Thm.

So next, we show $H^1(a,b) \hookrightarrow C^{0,\frac{1}{2}}[a,b]$.

i.e. $|u|_{0,\frac{1}{2}} \leq C\|u\|_{H^1}$.

$$|u(x) - u(y)| = \left|\int_y^x u'(t)\,dt\right|$$

$$\overset{\text{Hölder}}{\leq} |x-y|^{\frac{1}{2}} |u|_{1,2} \leq |x-y|^{\frac{1}{2}} \|u\|_{H^1}$$

Cor. For $0<\alpha<1/2$, $W^{1,p}(a,b) \overset{cpt}{\hookrightarrow} C^{0,\alpha}[a,b]$.

Lem. $C^\infty([a,b])$ is dense in $W^{1,p}(a,b)$, $p \in [1,\infty)$

$\{u \in C^\infty(a,b) \mid D^k u \text{ is uniformly conti. on } (a,b)\}$

(i.e. $D^k u$ can be extended uniquely to conti. func. on $[a,b]$)

Lem. (local approxi.)

$u \in W^{1,p}(a,b)$, $u_\varepsilon = \varphi_\varepsilon * u \in C^\infty(\mathbb{R}^1)$.

And for ε_0 small enough. We have:

$\|u - u_\varepsilon\|_{W^{1,p}(a+\varepsilon_0,\, b-\varepsilon_0)} \to 0$. $\forall \varepsilon < \varepsilon_0$

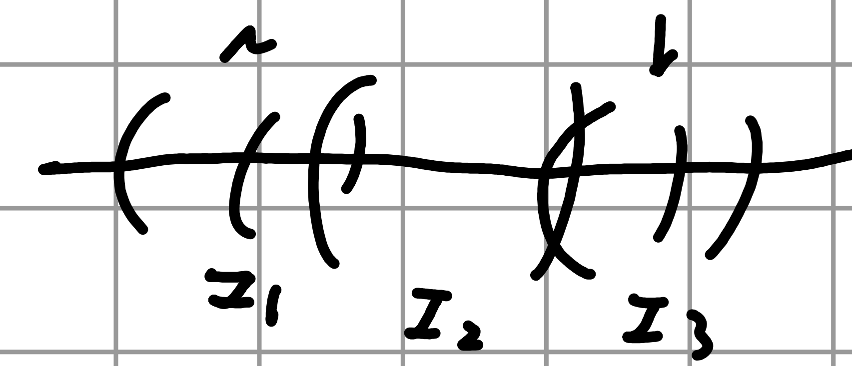
Pf: For $x \in (a+\varepsilon_0, b-\varepsilon_0)$, $\varphi_\varepsilon(x-\cdot) \in C_c^\infty(a,b)$. (Note: $\operatorname{supp} \varphi \subset [-1,1]$

$\Rightarrow \operatorname{supp}(\varphi_\varepsilon(x-\cdot)) = [x-\varepsilon, x+\varepsilon]$)

So: $(u')_\varepsilon(x) = (u_\varepsilon)'(x)$.

$\Rightarrow$ On $(a+\varepsilon_0, b-\varepsilon_1)$. We have:

$u_\varepsilon \xrightarrow{L^p} u$ and $(u_\varepsilon)' \xrightarrow{L^p} u'$.

<u>Pf</u>: Split it in:  $I_1 \cap I_3 = \emptyset$.

and $\bigcup_1^3 I_i \supset (a,b)$. Let $\varphi_i \in C^\infty(\mathbb{R}')$ is POU of $\bigcup_1^3 I_i$. $\mathrm{Supp}(\varphi_i) \subset I_i$.

For $u \in W^{1,p}(a,b)$. Set $u_i = u\varphi_i \in W^{1,p}(a,b)$

1) u_2 can be approxi. by Lem.

2) u_1: We shift δ to left:

$U_\delta^1(x) := u_1(x+\delta) \in W^{1,p}(a-\delta, b+\delta)$.

Since $\varphi_1 \equiv 0$ outside I_1.

Apply lem. again: $\exists (U_\delta^1)_\varepsilon \to U_\delta^1$ in $W^{1,p}(a,b)$. Also $\|U_\delta^1 - u_1\|_{L^p} \to 0$. $\delta \downarrow 0$

And shift commute with differential

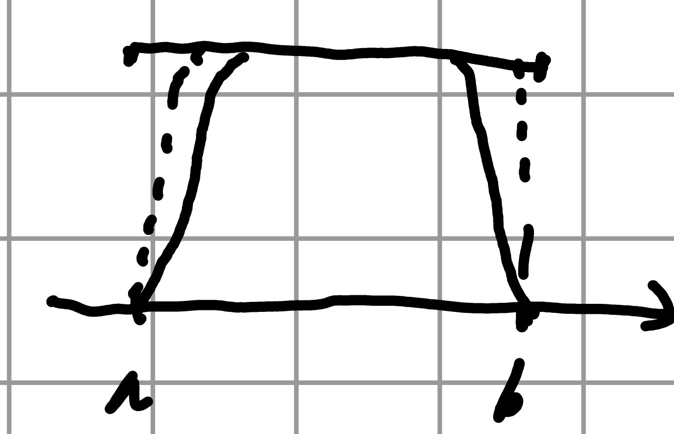
$\Rightarrow \|(U_\delta^1)' - u_1'\|_{L^p} \to 0$. $\Rightarrow U_\delta \xrightarrow{\|\cdot\|_{1,p}} u_1$

3) We similarly approxi u_3 as 2)

<u>Cor.</u> $C^k([a,b])$ is dense in $W^{1,p}(a,b)$.

<u>Rmk</u>: $C_c^\infty(a,b) \overset{\text{dense}}{\not\hookrightarrow} W^{1,p}(a,b)$. But:

$\overline{C_c^\infty(a,b)}^{W^{1,p}(a,b)}$ is proper subspace of $W^{1,p}(a,b)$ denoted by $W_0^{1,p}(a,b)$. So: it's Banach for $1 \le p \le \infty$. And sep. for $1 \le p < \infty$.



<u>e.g.</u> $u \equiv 1 \in W^{1,p}(a,b)$

But when approxi. its deri. at $[a,b]$ by C_c^∞. $\|u_n\|_{1,p}$ explodes.

<u>Cor.</u> $u \in W^{1,1}(a,b) \Rightarrow u(x) - u(y) = (x-y)\int_0^1 u'(y+t(x-y))$

<u>Pf</u>: By $W^{1,1} \hookrightarrow C[a,b]$. So: $u(x)$ is well-def.

Use Lem. Approxi. u by $u_n \in C^\infty$.

<u>Lem</u>. X, Y. Banach. If $\varphi: X \overset{d}{\hookrightarrow} Y$. Then:

i) $Y^* \hookrightarrow X^*$. ii) X is reflexive $\Rightarrow Y^* \overset{d}{\hookrightarrow} X^*$

<u>Pf</u>: i) Set $\varphi^*: Y^* \to X^*$ by $\langle \varphi^*(y^*), x\rangle = \langle y^*, \varphi(x)\rangle$. dual operator of φ

Note: φ^* is injective $\Leftrightarrow$ $R(\varphi)$ is dense in Y.

ii) For $x^{**} \in X^{**}$. if: $\forall y^* \in Y^*$.

$$\langle x^{**}, \varphi^*(y^*)\rangle \overset{\text{def}}{=} \langle \varphi^*(y^*), x\rangle$$
$$= \langle y^*, \varphi(x)\rangle = 0.$$

if $\varphi(x) \neq 0. \Rightarrow \exists\, y_0^* \in Y^*$. s.t.

$\langle y_0^*, \varphi(x)\rangle \neq 0$. by Hahn-Banach Thm.

$\Rightarrow$ Contradict! So: $\varphi(x) = 0 \Rightarrow x = 0 \Rightarrow x^{**} = 0$

<u>Thm.</u> (Charac. of $W_0^{1,p}(a,b)$)

$$W_0^{1,p}(a,b) = \{ u \in W^{1,p}(a,b) \mid u(a) = u(b) = 0 \}.$$

<u>Rmk</u>: It's kind of Dirichlet condition.

<u>Pf</u>: i) For $(\varphi_n) \subset C_c^\infty(a,b) \xrightarrow{\|\cdot\|_{1,p}} u \in W_0^{1,p}(a,b)$

With $W^{1,1} \hookrightarrow C[a,b]$. So:

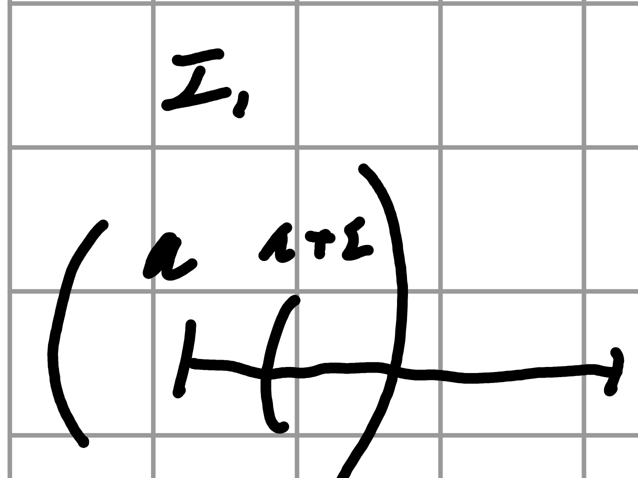
$\|\varphi_n - u\|_{C[a,b]} \le C \|\varphi_n - u\|_{W^{1,p}} \to 0.$

$\Rightarrow |\varphi_n(a) - u(a)| \xrightarrow{n\to\infty} 0$ (also for $x = b$)

ii) For $u \in W^{1,p}(a,b)$ $u(a) = u(b) = 0$.

Consider $w \in C^\infty(\mathbb{R}^1)$. s.t. $w(x) = 0$ on

$\{|x| \le 1\}$ and $w(x) = 1$ on $\{|x| \ge 2\}$.



Set $u_\varepsilon(x) := u(x)\, w\left(\frac{x-a}{\varepsilon}\right) \in W^{1,p}(a,b)$

$$u_\varepsilon(x) \begin{cases} = 0 & \text{on } \{a \le x \le a+\varepsilon\} \\ = u(x) & \text{on } \{x > a+2\varepsilon\} \end{cases}$$

Denote $W_{\varepsilon,a}(x) = W(\frac{x-a}{\varepsilon})$

$$\|u-u_\varepsilon\|^p_{W^{1,p}(a,b)} = \int_a^b |u|^p |1-W_{\varepsilon,a}|^p + \int_a^b |u'-u_\varepsilon'|^p$$

$$\leq A_1+A_2+A_3 \triangleq \int_a^{a+2\varepsilon} |u|^p + |u'|^p + \int_{a+\varepsilon}^{a+2\varepsilon} |u|^p |W_{\varepsilon,a}'|^p dx$$

Note $A_1, A_2 \to 0$ $(\varepsilon\to 0)$. For A_3:

First, note we have: $|W_{\varepsilon,a}'| \leq C/\varepsilon$.

And $u(x) = \int_a^x u'(t)dt$. a.e. by $u(a)=0$

$$\text{So: } A_3 \lesssim \varepsilon^{-p}\int_a^{a+2\varepsilon} \left|\int_a^x |u'| dt\right|^p dx$$

$$\overset{\text{Hölder}}{\lesssim} \varepsilon^{-p}\int_a^{a+2\varepsilon} |x-a|^{p/p'} \int_a^x |u'|^p dt\, dx.$$

$$\overset{\text{Fubini}}{\lesssim} \varepsilon^{-p}\varepsilon^{p/p'}\int_a^{a+2\varepsilon} |u'|^p \int_t^{a+2\varepsilon} dx\, dt$$

$$\lesssim \varepsilon^{-p+p/p'+1}\int_a^{a+2\varepsilon} |u'|^p$$

$$= \int_a^{a+2\varepsilon} |u'|^p dt \to 0 \ (\varepsilon\to 0)$$

Then smoothize: $u_\varepsilon^\delta = \varphi_\delta * u_\varepsilon$.

$\Rightarrow$ for ε_0 small. $u_\varepsilon^\delta = 0$ on $[a\leq x\leq a+\varepsilon_0]$

And split u into three u_i on I_i as before. Then we can approxi. u_1 by C_c^∞ so u_2, u_3. Then $\exists u^\varepsilon \in C_c^\infty \to u$.

Lem. (Poincaré inequi.)

$\exists C > 0$. s.t. $\|u\|_{0,p} \leq C|u|_{1,p}$. $\forall u \in W_0^{1,p}(a,b)$.

Pf: $u \in W_0^{1,p}(a,b) \Rightarrow u \in AC[a,b]$.

So $u(x) = \int_a^x u'(t)dt$. $(u(a) = 0)$.

$$\|u\|_{0,p}^p = \int_a^b \left|\int_a^x u'(t)dt\right|^p dx \overset{\text{Hölder}}{\leq} \int_a^b |x-a|^{p/2} \|u'\|_{0,p}^p$$

Rmk: i) Note nonzero const. $\notin W_0^{1,p}(a,b)$. So the inequi. doesn't hold.

ii) $C = \frac{b-a}{2}$ is optimal for $p=2$.

iii) On $W_0^{1,p}(a,b)$, $\|\cdot\|_{W^{1,p}} \sim |\cdot|_{1,p}$. Next we equip $W_0^{1,p}(a,b)$ with $|\cdot|_{1,p}$.

Thm. (Rellich)

$H^1(a,b) \overset{c}{\hookrightarrow} L^2(a,b)$.

Pf: $H^1(a,b) \hookrightarrow L^2(a,b)$ is trivial. So:

We only need to prove its cptness.

For $(u_n) \subset H^1(a,b)$ bdd by M.

$$\int_a^b |u_n(x+h) - u_n(x)|^2 = \int_a^b \left|\int_x^{x+h} u_n'\right|^2 dx$$

$$\overset{\text{Hölder}}{\leq} \int_a^b h \int_x^{x+h} |u_n'|^2 \leq M^2(b-a)h$$

Apply L^p-Ascoli Thm. We have it.

Rmk: Similarly, $W^{1,p}(a,b) \overset{c}{\hookrightarrow} L^p(a,b)$. $\forall p > 1$.

<u>Def</u>: For $p<\infty$. We denote $W^{-1,p'}(a,b) := W^{1,p}(a,b)^*$

And denote $(H_0^1(a,b))^*$ by $H^{-1}(a,b)$

<u>Rmk</u>: "-1" means negative derivative i.e. it's integration.

<u>e.g.</u> $\langle f, v\rangle := \int_0^1 \frac{1}{x} v(x)dx \Rightarrow f \in H^{-1}(0,1) / H^1(0,1)^*$.

Take $v\equiv 1 \Rightarrow \langle f, 1\rangle = \int_0^1 1/x = \infty$. However:

$|\langle f, v\rangle| \overset{\text{Hölder}}{\leq} \int_0^1 \frac{1}{x} (\int_0^x 1)^{\frac{1}{2}} |v|_{1,2} \leq 2\|v\|_{1,2}$. $\forall v \in H_0^1$

So: $\langle f, \cdot\rangle \in H^{-1}(0,1)$