Diff. Equations

(1) <u>Variational formulation for BVP:</u>

Consider $\begin{cases} Lu(x) := -u''(x) + c(x)u'(x) + d(x)u(x) = f(x) \\ u(a) = u(b) = 0. \quad x \in [a,b]. \end{cases}$

We want to get its weak formulation:

1) multiply both sides with suitable test func.

2) Integrate both sides:

$$\int_a^b (-u''v + cu'v + duv)\,dx = \int_a^b fv\,dx. \quad v \in V$$

3) Partial integrate on highest order part.

$$\int_a^b (u'v' + cu'v + duv)\,dx - u'v\Big|_a^b = f(x).$$

We want the sol. space = test func space V.

$\Rightarrow$ Assume $v(a) = v(b) = 0$.

Also $u'v' \in L^1 \Rightarrow u, v \in H_0^1$. i.e. space $V = H_0^1$.

And assume $c, d \in L^\infty(a,b)$ to let the integral makes sense. (It can be weakened: $c \in L^2$. $d \in L^1$. $f \in (H_0^1(a,b))^*$ since $H^1(a,b) \hookrightarrow C[a,b]$)

Rmk: We can assume V is smooth enough that assures it can be recovered to original equation. e.g. $V = C_c^\infty$. Then: by FTCV: the weak sol. is true sol.

4) For $\int fv$ part: We assume $f \in L^2$.

$\Rightarrow$ We get $\int_a^b (cu''v'' + cu'v + kuv) = \int_a^b fv$. $v \in H_0^2$.

Denote: $a(u,v) := \int_a^b (cu''v'' + cu'v + kuv) : V \times V \to \mathbb{R}^1$

$\langle \tilde{f}, v \rangle := \int_a^b fv$. $\tilde{f} \in V^*$. $v \in V$.

Rmk: By Poincaré & Hölder inequi.:

$\langle \tilde{f}, v \rangle \leq \|f\|_{0,2} |v|_{1,2}$.

$\Rightarrow$ it's BLO on H_0^2. So $V^* = H^{-2}$.

Also, $a(u,\cdot)$ is also BLO on H_0^2 for $\forall u \in V = H_0^2$. $\Rightarrow a(u,\cdot) \in H^{-2}$.

Set $\langle Au, v \rangle \triangleq a(u,v)$. $A: V \to V^*$. $\forall u, v \in V$.

Then it becomes operator eq.: $\langle Au, v \rangle = \langle \tilde{f}, v \rangle$.

i.e. $Au = \tilde{f}$ in V^*.

e.g. i) On $(-1,1)$. Let $c, k \in C^\infty(-1,1)$. $V = H_0^1$.

$f = I_{\{x \geq 0\}} - I_{\{x < 0\}}$.

So: $|\langle \tilde{f}, v\rangle| \lesssim \|v\|_{0,2} \overset{\text{poincare}}{\lesssim} \|v\|_{H_0^1}$. BLO on V.

ii) On $(-1, 1)$. $f = \delta_0$. Dirac measure.

For $V = H_0^1$. Note $H_0^1 \hookrightarrow C[-1,1]$. So key:

$\langle \tilde{\delta_0}, v\rangle = v(0)$. for $v \in H_0^1$ is well-def. And

$|\langle \tilde{\delta_0}, v\rangle| \leq |v(0)| \leq \|v\|_{C[-1,1]} \lesssim \|v\|_{1,2}$

iii) Different from i), ii). We replace $u(a) = u(b) = 0$ by $u'(a) = \alpha$. $u'(b) = \beta$. (Neumann)

And its variation formulation becomes

$$-\int u'' v + \square = \int u' v' + \square - u'(b) v(b) + u'(a) v(a)$$

$$= \int f v$$

For $c, d \in L^\infty$. $V = H^1$. We set

$$a(u, v) := \int u' v' + c u' v + d u v = \langle \tilde{f}, v\rangle$$

$$:= \int f v + \beta v(b) - \alpha v(a).$$

<u>Rmk</u>: i) Note for $\alpha = \beta = 0$. We have formulation

$a(u, v) = \int f v$ on larger space H^1.

ii) $V = \{ v \in H^1 \mid u(a) = \alpha, u(b) = \beta \}$ isn't

LS. But it's Complete in H^1:

For (u_n) Cauchy in $H^1 \Rightarrow \exists u \in H^1$

s.t. $u_n \xrightarrow{H^1} u$. Desired:

$|u(a) - \alpha| \le \|u - u_n\|_{C[a,b]} \le \|u - u_n\|_{H^1} \to 0$

since $H^1(a,b) \hookrightarrow C[a,b]$

Consider general BVP $\begin{cases} -u'' + cu' + ku = f. \\ u(a) = \alpha. \quad u(b) = \beta. \end{cases}$

$\Rightarrow$ Idea: Transfer inhomogeneous Dirichlet boundary

condition to homogeneous case.

Set $r \in C^2$ s.t. $r(a) = \alpha$. $r(b) = \beta$. And consider

$\tilde{u} = u - r \in H_0^1(a,b)$.

$a(\tilde{u}, v) = a(u, v) - a(r, v) = \langle \tilde{f}, v \rangle - a(r, v)$

$=: \langle \hat{f}, v \rangle$. reduce to homo case!

(2) Lax-Milgram:

Def: $a : V \times V \to \mathbb{R}^1$. bilinear, $A : V \to V^*$. linear.

i) a is bdd if $\exists \beta > 0$. $|a(u,v)| \overset{\forall u,v}{\le} \beta \|u\| \|v\|$.

ii) a is strictly positive if $a(u,u) > 0$. $\forall u$

(For A: we define $\langle Au, u \rangle > 0$. $\forall u$)

iii) a is strongly positive if $\exists c > 0$. s.t.

$a(u,u) \ge c\|u\|^2$. $\forall u \in V$.

(For A: $\langle Au, u\rangle \geq c\|u\|^2$. $\forall u$)

Rmk: ii) $\not\Rightarrow$ iii) if $\lim V = \infty$. (When $\lim V < \infty$. it's true) e.g. $(e_n) \subset \ell^2$ o.n.b.

Set $a(e_n, e_m) = 2^{-m}\delta_{mn}$. $a(e_m) \to 0$.

iv) a is symmetry if $a(u,v) = a(v,u)$.

(For A: $\langle Au, v\rangle = \langle Av, u\rangle$)

Rmk: On BVP above. a is defined to be symmetric if $c=0$.

Thm. (Lax - Milgram)

V is Banach. $A: V \to V^*$ strongly positive ($\langle Au, u\rangle \geq \mu\|u\|^2$) BLO. $\Rightarrow$ A is bijection.

Cor. If $a(\cdot,\cdot)$ linear. bdd. strongly positive.

Then: for $f \in V^*$. $a(u,v) = \langle f, v\rangle$. $\forall v \in V$. has unique sol. u.

Pf: $Au \triangleq a(u, \cdot): u \in V \mapsto V^*$ satisfy cond.

Pf: Define $(u,v)_A := \frac{1}{2}(\langle Au, v\rangle + \langle Av, u\rangle)$ is bilinear and symmetric. $\|u\|_A := (u,u)_A^{\frac{1}{2}}$.

$\Rightarrow \|\cdot\|_A \sim \|\cdot\|$ by bdd & strong positive

So: $(V, (\cdot,\cdot)_A)$ is Hilbert space.

By Riesz: $\exists I: V^* \xrightarrow{\sim} V$ isometric isomor.

s.t. $(If, u)_A = \langle f, u \rangle$. $\forall f \in V^*, u \in V$.

Consider $\phi(u) = u + zI(f - Au)$. $z > 0$. So:

$\forall f \in V^*$: u is sol. of $Au = f$. $\Leftrightarrow \phi(u) = u$.

Note $\|\phi(u) - \phi(v)\|^2 = \|u-v\|^2 + z^2\|Au - Av\|^2 + \langle\cdot,\cdot\rangle$

$\le \|u-v\|^2 + z^2\|A\|^2\|u-v\|^2 - 2z\mu\|u-v\|^2$.

Choose z small enough $\Rightarrow \phi$ is contraction.

Then apply Banach fixed pt Thm.

<u>Cor.</u> $A^{-1}: V^* \to V$ exists on above and

it's strongly positive BLO.

<u>e.g.</u> Consider $\begin{cases} -u'' + cu' + du = f. & \text{on } (a,b) \\ u(a) = u(b) = 0. & \text{where } c, d \in C^\infty. \end{cases}$

$\Rightarrow \langle Au, v \rangle \stackrel{\Delta}{=} \int u'v' + cu'v + duv$

$= \langle \tilde{f}, v \rangle \stackrel{\Delta}{=} \int fv$. So A is BLO:

By $|\langle Au, v\rangle| \le C(1 + C\|c\|_\infty + C^2\|d\|_\infty)|u|_{1,2}|v|_{1,2}$.

For ~~strongly~~ positive:

i) $c \equiv d \equiv 0$. $\Rightarrow$ holds is strongly positive

ii) $c=0$. $k=-1$. $\langle Au, u\rangle = |u|_{1,2}^2 - \|u\|_{0,2}^2$

By poincaré $\|u\|_{0,2} \leq (b-a)\pi^{-1}|u|_{1,2}$.

So strongly positive prop. depend on (a,b).

e.g. On $(0,\pi)$. take $u=\sin(x)$. Then:

$\langle Au, u\rangle = 0$. it's not strongly positive.

Thm. For $Au=f$ with Dirichlet boundary. $V = H_0^1(a,b)$. $f\in V^*$. $c, c', k \in L^\infty(a,b)$. s.t.

$\exists \underline{\alpha} > -\pi^2/(b-a)^2$. $k(x) - \frac{1}{2}c'(x) \geq \underline{\alpha}$. $\forall x$. Then:

there exists unique sol. for the equation

Pf: Note $\int cu'u = -\int c'\cdot \frac{1}{2}u^2$

$$\Rightarrow \langle Au, u\rangle = |u|_{1,2}^2 + \int (k - \tfrac{c'}{2})u^2$$

e.g. For $f\in H^{-1}(a,b)$. $p, q \in C[a,b]$ satisfies:

$p(x)\geq \mu$. $q(x) > -\pi^2\mu/(b-a)^2$. Then:

$$\begin{cases} -(p(x)u'(x))' + q(x)u(x) = f(x). & x\in(a,b) \\ u(a)=u(b)=0. \end{cases}$$

Pf: Directly apply Lax-Milgram Thm.

Rmk: (Regularity Theory).

Note for $\begin{cases} -u''=f & \text{on } (a,b) \\ u(a)=u(b)=0 \end{cases}$

we can only require $f \in H^{-1}$. Then the sol. u will fall on $H_0^{-1+2} = H_0^1$

For $f \in L^2$. Then $u \in H_0^{0+2} = H_0^2$.

For $f \in H^k$. $\overset{?}{\Rightarrow}$ $u \in H^{k+2}$.

(3) Nonlinear Case:

Def: $A: V \to V^*$ is strongly mono. if $\exists \mu > 0$. st.

$\langle Au - Av, u - v \rangle \geq \mu \|u - v\|^2$. $\forall u, v \in V$.

Thm. (Zarantonello for nonlinear case)

For V Hilbert space. $A: V \to V^*$ is Lip. conti. and strongly mono. $\Rightarrow$ A is bijection. Besides. A^{-1} is Lip-conti. and strongly mono.

Pf: Consider $\Phi: V \to V$. $\Phi(v) = v + z I(Au - f)$ for $z > 0$ as before.

$\|\Phi(u) - \Phi(v)\|^2 =$

$\|u - v\|^2 - 2z \langle Au - Av, u - v \rangle + \|Au - Av\|^2$

Proceed as before: We can find $z > 0$ st. Φ is contraction.

e.g. $\begin{cases} -(\varphi(|u'|)u')' + cu' + \lambda u = f & \text{on } (a,b) \\ u(a) = u(b) = 0. \end{cases}$

where $\varphi \in C(\mathbb{R}^{\geq 0}, \mathbb{R})$. Satisfies: $\exists M, m > 0$.

i) $|\varphi(x)| \leq M$. ii) $\varphi(s)s - \varphi(t)t \geq m(s-t)$. $\forall s \geq t \geq 0$

iii) $|\varphi(s)s - \varphi(t)t| \leq M|t-s|$. $\forall t, s \geq 0$.

So $t\varphi(t)$ is Lip-conti. $\uparrow$. And we assume

$c, c', \lambda \in L^\infty$. $f \in L^2$.

$\Rightarrow$ Variation reformulation:

$$\int_a^b \varphi(|u'|)u'v' + cu'v + \lambda uv = \int fv \overset{\Delta}{=} \langle \tilde{f}, v \rangle$$

Choose $V = H_0^1(a,b)$. Denote LHS $= \langle Au, v \rangle$.

We claim: A is Lip-conti. & strongly mono.

i) For Lip-conti.

Only consider $\int (\varphi(|u'|)u' - \varphi(|w'|)w')v'$

If $u', w' \geq 0$. Then: by prop. of φ.

$\Rightarrow$ it $\overset{\text{iii)}}{\leq} M \int |u' - w'||v'| \lesssim |u-w|_{1,2} |v|_{1,2}$.

For $u' \geq 0$, $w' < 0$. (Also $u' < 0$, $w' \geq 0$) Then:

it $\leq \int |\varphi(u')u' + \varphi(-w')(-w')| |v'|$.

$\overset{i)}{\leq} \int m(u'-w')|v|$

2) For strongly mono. :

$\langle Au - Aw, u-w \rangle = \int (\varphi(|u'|)u' - \varphi(|w'|)w')(u-w)'$

$+ c\left(\frac{1}{2}(u-w)^2\right)' + \lambda(u-w)^2 \triangleq A+B+C.$

$B = \int -\frac{c'}{2}(u-w)^2$. For A:

If $u' \geq w' \geq 0$. $A \overset{iii)}{\geq} \int m(u'-w')^2$.

Note $\|u-w\|_{0,2}^2 \leq \frac{(b-a)^2}{\pi^2}|u-w|_{1,2}^2$

So: when $\lambda - \frac{c'}{2} \geq \underline{\lambda} > -m\pi^2/(b-a)^2$ on (a,b)

We have it's strongly mono.