Brownian Interlacement.

(1) Brownian capacity:

① Green func.

Def: i) $G(x) =: \Gamma(\frac{d}{2}-1)|x|^{2-d}/2\pi^{d/2}$ is green func. for SBM in $\mathbb{R}^d$. $d \geq 3$.

ii) Denote $g(x)$ is green func. for SRW in $\mathbb{Z}^d$. $d \geq 3$.

Prop. $g(x) \sim d\,G(x)$. $(x \to \infty)$

Cor. Denote $\mathbb{L}_N = \frac{1}{N}\mathbb{Z}^d \subset \mathbb{R}^d$. $N > 1$.

$g_N(y) = \frac{1}{d} N^{d-2} g(Ny)$. Then: $\forall \gamma > 0$.

$$\lim_{N\to\infty} \sup_{|y| \geq \gamma} |g_N(y) - G(y)| = 0.$$

prop. $P_t(y, y') =: (2\pi t)^{-\frac{d}{2}} e^{-|y-y'|^2/2t}$ trans. prob. of SBM in $\mathbb{R}^d$. Then: $G(x, x') = \int_0^\infty P_t(x, x')\,dt$

② Equilibrim measure:

Def: For $K \subset \mathbb{R}^d$. $V: \mathbb{R}^d \to \mathbb{R}^1$. killing rate

Q_z is law of BM (Z_t) killed at

rate V when leaving U. $\tau(w, \zeta) =: \tau_u(w)$

$\wedge \inf\{s \geq 0 \mid \int_0^s V(Z_u(w))\,du \geq \zeta\}$. death time.

Rmk: i) $V \in K^d_{loc} =: \{ f \in B_{\mathbb{R}^d}.\ \lim_{t\to 0} \sup_x \mathbb{E}_x(\int_0^t |f(Z_s)|ds) = 0 \}$

is required following

ii) Set $\mathbb{P}_z$ is law of BM starting at

z. $\hat{\mathbb{P}}_z =: \mathbb{P}_z \otimes e^{-\zeta} d\zeta$. $\zeta =: \inf\{s \geq 0 \mid Z_s = \partial\}$

$$Q_z(Z_t \in A) = \hat{\mathbb{P}}_z(Z_t \in A.\ \tau > t)$$

$$= \hat{\mathbb{P}}_z(Z_t \in A.\ \tau_u > t,\ \int_0^t V < \zeta)$$

Thm. (equilibrim Problem)

$\psi_\varepsilon(z)dz =: Q_z(0 < L_K \leq \varepsilon)dz/\varepsilon$. $L_K =:$

$\sup\{t > 0 \mid Z_t \in K\}$. last entrance time.

Then. $\psi_\varepsilon(z)dz \xrightarrow{v}$ a finite measure supports

on K. Moreover, the measure has no atom if $d \geq 2$

Rmk: We denote such measure by $e_K(dz)$.

called equilibrim measure of K.

Thm. (last exit formula)

$$\mathbb{P}_\eta(L_K > 0.\ L_K \in dt.\ X_{L_K} \in dz) = p_t(\eta, z)\, e_K(dz)dt$$

Cor. $\mathbb{P}_\eta(L_K > 0.\ X_{L_K} \in dz) = G(\eta, z)\, e_K(dz)$

Thm. $\mathrm{cap}_{\mathbb{R}^d}(K) =: \int_K e_K(dz) = \inf\{\frac{1}{2}\int |\nabla\phi|^2 \mid$

$\mathrm{supp}(\phi) \subset K,\ \phi \geq 1 \text{ on } K\}$. capacity of K.

Thm (Variational Characterization)

$K \underset{cpt}{\subseteq} \mathbb{R}^d$. $\mathcal{E}(\mu) =: \int_{K\times K} G(x,y)\, \mu(dx)\mu(dy)$.

Then: $\mathrm{cap}(K) = 1 / \inf \{ \mathcal{E}(\mu) \mid \mu \text{ is p.m. on } K \}$

prop. i) $\mathrm{cap}_{\mathbb{R}^d}(A \cup B) \le \mathrm{cap}_{\mathbb{R}^d}(A) + \mathrm{cap}_{\mathbb{R}^d}(B)$

ii) $\mathrm{cap}_{\mathbb{R}^d}(A) \le \mathrm{cap}_{\mathbb{R}^d}(B)$ if $A \subseteq B$.

iii) $\mathrm{cap}_{\mathbb{R}^d}(\cdot)$ is invariant under translation.

prop. $K \underset{cpt}{\subseteq} \mathbb{R}^d$ with positive volume. We have:

$$\frac{|K|}{\sup_{z\in K} \int_K G(z,dy)} \le \mathrm{cap}_{\mathbb{R}^d}(K) \le \frac{|K|}{\inf_{z\in K} \int_K G(z,dy)}$$

Pf: Note for $z \in K$:

$$\int_{K\times K} G(z,y)\, e_K(dy)\, dz = \int_K \mathbb{P}^z(H_K < \infty)$$

$$= |K|. \quad \le \sup \square \cdot \mathrm{cap}_{\mathbb{R}^d}(K) \text{ or } \ge \inf \square .$$

Cor. $\mathrm{cap}_{\mathbb{R}^d}(B_R) = C R^{d-2}$.

prop. (Sweeping id.)

$e_K(\cdot) = \mathbb{P}_{e_B}(H_K < \infty, X_{H_K} \in \cdot)$ for $K \subseteq B$

cpt set in $\mathbb{R}^d$.

(2) Construction:

① **Def**: i) For $d \geq 3$. $W =: \{ f \in C(\mathbb{R}, \mathbb{R}^d) \mid f \xrightarrow{x \to \pm\infty} \infty \}$.

$W^+ =: \{ f \in C(\mathbb{R}^{\geq 0}, \mathbb{R}^d) \mid f \xrightarrow{x \to +\infty} \infty \}$ endowed with σ-algebra $\mathcal{W}$, $\mathcal{W}^+$, generated by canonical process.

Set $W^* = W / \underset{shift}{\sim}$. $W_B^k = W \cap \{ H_B(f) = k \}$.

ii) Q_B is a measure on W_B^0, defined by:

$Q_B((X_{-t})_{t \geq 0} \in A', X_0 \in dy, (X_t)_{t \geq 0} \in A) = e_B(dy) \, \mathbb{P}_y^B(A') \, \mathbb{P}_y(A)$, for $A, A' \in \mathcal{W}^+$.

where $\mathbb{P}_y^B(\cdot) = \mathbb{P}_y(\cdot \mid \tilde{H}_B = \infty)$, $y \in B$, closed ball.

Rmk: It's a local chart of the intensity measure on W_B^0.

Lemma. If B, B' are closed balls, $B \subset \operatorname{int} B'$.

Then $\theta_{H_B} \circ (1_{\{H_B < \infty\}} Q_{B'}) = Q_B$.

Def: $\forall K$ cpt subset of $\mathbb{R}^d$, $Q_K =: \theta_{H_K} \circ (1_{\{H_K < \infty\}} Q_B)$ for $K \subset B$, B is closed ball.

Rmk: It's well-def: if $K \subset B_1$, $K \subset B_2$, set $B' \supseteq B_1 \cup B_2$ closed ball. Then:

$\theta_{H_K} \circ (1_{\{H_K < \infty\}} Q_{B_1}) = \theta_{H_K} (1_{\{H_K < \infty\}} \theta_{H_{B_1}} (1_{\{H_{B_1} < \infty\}} \cdot Q_{B'}))$

$= \theta_{H_K} (1_{\{H_K < \infty\}} Q_{B'})$

Thm. There exists a unique σ-finite measure ν on $(W^*, \mathcal{W}^*)$. s.t. $\forall K \underset{cpt}{\subseteq} \mathbb{R}^d$. we have:

$I_{W_K^*} \nu = \pi^* \circ Q_K$. $\pi^* : W \to W^*$. canonical.

Pf: Set $(K_n) \uparrow \mathbb{R}^d$. seq of cpt sets.

$W^* = \bigcup_n W^*_{K_n} \Rightarrow$ we get uniqueness.

Next. check: $\pi^* \circ Q_K = \pi^* (I_{W_K} Q_{K'}) = I_{W_K^*} \pi^* Q_{K'}$.

for $\forall K \subset K'$. cpt. sets.

It follows from: $\begin{cases} Q_K = \theta_{H_K} (I_{W_K} Q_B) \\ Q_{K'} = \theta_{H_{K'}} (I_{W_{K'}} Q_B) \end{cases}$

for $K, K' \subseteq B$. $\Rightarrow I_{W_K} Q_{K'} = \theta_{H_K} (I_{W_K} Q_B)$

Then. define: $I_{W^*_{K_n}} \nu =: \pi^* \circ Q_{K_n}$.

which doesn't depend on choice of (K_n)

Let $K_1 = K$. it satisfies the condition.

prop. i) $\check{\nu} = \nu$. inverse-invariant.

ii) ν is invariant under $\tau_\eta : W^* \mapsto W^* - \eta$.

iii) ν is invariant under $R : W^* \overset{iso.}{\mapsto} RW^*$. linear.

iv) $S_\lambda \circ \nu = \lambda^{2-d} \nu$ for scaling $S_\lambda : W^* \mapsto \lambda W^*(\frac{\cdot}{\lambda^2})$

Pf: Denote $\mathbb{P}^t_{x,\eta} =: \mathbb{P}_x(\cdot \mid Z_t = \eta) \, I[Z_t \in \eta)$.

$Q_B((X_{-t})_{t\geq 0} \in A'. (X_s)_{0\leq s\leq L_B} \in \cdot, (X_{L_B+s})_{s\geq 0} \in A)$

$= \int_0^\infty \int \mathbb{P}^B_{\eta'}(A') \, e_B(d\eta') \, \mathbb{P}^t_{\eta',\eta}(\cdot) \, \mathbb{P}^B_\eta(A) \, \mathbb{P}_{\eta'}(L_B \in dt, X_{L_B} \in d\eta) dt$

$\overset{last-exit}{=} \int_0^\infty \int \mathbb{P}^B_{\eta'}(A') \, e_B(d\eta') \, \mathbb{P}^t_{\eta',\eta}(\cdot) \, \mathbb{P}^B_\eta(A) \, e_B(d\eta) \, p_t(\eta',\eta) \, dt$

Note $p_t(\cdot,\cdot)$ is sym. $\check{\mathbb{P}}^t_{y,y'} = \mathbb{P}^t_{y',y}$.

$\Rightarrow$ We obtain i). ii). iii).

Besides, $\lambda^{2-d} = \lambda^{-d}$ (position scaled) $\cdot \lambda^2$ (time scale)

<u>Thm.</u> Set $\varphi : w^* \in W_K^* \longmapsto w^{*,K,+} =: (w(H_K+t))_{t\geq 0}$

$\Rightarrow \varphi \circ (I_{W_K^*} \nu) = \mathbb{P}_{e_K}$

<u>pf:</u> $Q_K((X_t)_{t\geq 0}) = \theta_{H_K}(I[H_K<\infty] Q_B((X_t)_{t\geq 0}))$

$= \int e_B(dy)\, \mathbb{P}_y(\cdot)$

$\overset{\text{sweeping}}{\underset{\text{id.}}{=}} \mathbb{P}_{e_K}(\cdot)$.

Combine with $I_{W_K^*} \nu = Q_K$.

③ <u>Def</u>: i) $\Omega = \{ \omega = \sum_{i\geq 0} \delta_{(w_i^*, \alpha_i)} \mid (w_i^*, \alpha_i) \in W^* \times \mathbb{R}_{\geq 0}$.

$\omega(W_K^* \times [0,\alpha]) < \infty$. $\forall K$. cpt. $\alpha > 0 \}$.

the space of point measure on $W \times \mathbb{R}_+$.

ii) Endow Ω with $\mathcal{A} = \sigma\{\omega \overset{\varphi}{\longmapsto} \omega(B),\ \forall B \in W^* \times \mathcal{B}_{\mathbb{R}}\}$

iii) $\mathbb{P}$ is law on $(\Omega, \mathcal{A})$. of PPP with intensity $\nu \otimes d\alpha$ on $W^* \otimes \mathbb{R}$.

<u>prop.</u> $\mu_{K,\alpha}(\omega) = \sum_{i\geq 1} I[\alpha_i \leq \alpha,\ w_i \in W_K^*]\, \delta_{w_i^{*,K,+}}$ for $\omega = \sum_{i\geq 1} \delta_{(w_i^*, \alpha_i)}$

is PPP on W^+ with intensity $\alpha \mathbb{P}_{e_K}$.

Prop. $\mathbb{P}(\cdot)$ is invariant on Ω. under:

i) $w = \sum \delta(w_i^*, \alpha_i) \mapsto \sum \delta(\check{w}_i^*, \alpha_i)$

ii) $w \mapsto \sum \delta(w_i^* - y, \alpha_i)$ iii) $w \mapsto \sum \delta(R w_i^*, \alpha_i)$

iv) $w \mapsto \sum \delta(S_\lambda w_i^*, \lambda^{-d+2} \alpha_i)$

Pf: It follows from the prop. of ν.

Def: i) $\mathcal{I}_r^\alpha(w) =: \bigcup_{\alpha_i \le \alpha} \bigcup_{s \in \mathbb{R}'} \overline{B}(w_i(s), r)$. Brownian interlacement at level α. with radius r.

$\mathcal{I}_0^\alpha(w)$ is call Brownian fabric at level α.

ii) Denote Q_r^α is law of $\mathcal{I}_r^\alpha$ on $(\Sigma, \mathcal{F})$

$=: (\{A \underset{\text{close}}{\subset} \mathbb{R}^d\}, \sigma(\{F \in \Sigma \mid F \cap K = \emptyset,\ \exists K \text{ cpt}\}))$

Prop. i) $Q_r^\alpha(\{F \in \Sigma \mid F \cap K = \emptyset\}) = \mathbb{P}(\mathcal{I}_r^\alpha \cap K = \emptyset)$

$= e^{-\alpha \,\mathrm{cap}_{BM}(K^r)}$. for $K \underset{\text{cpt}}{\subset} \mathbb{R}^d$.

ii) $\mathcal{I}_r^\alpha \overset{\mathbb{P}}{\sim} \mathcal{I}_r^\alpha + y$ iii) $R(\mathcal{I}_r^\alpha) \overset{\mathbb{P}}{\sim} \mathcal{I}_r^\alpha$.

iv) $\lambda \mathcal{I}_r^\alpha \overset{\mathbb{P}}{\sim} \mathcal{I}_{\lambda r}^{\lambda^{2-d}\alpha}$.

v) $\mathcal{I}_0^\alpha$ is a.s. connected when $d = 3$.

$\mathcal{I}_0^\alpha$ is a.s. disconnected when $d \ge 4$. $\alpha > 0$.

Prop. For $d \ge 3$. $\mathcal{I}_r^\alpha$ is a.s. closed. $\forall \alpha, r > 0$.

Pf: Prove: $\forall B$ closed ball. $B \cap \mathcal{I}_r^\alpha$ is close.

i) $\mathcal{I}_r^\alpha \cap B \sim \mathrm{POI}(\alpha \nu(W_{B^r}))$. $\Rightarrow \bigcup_{\alpha_i \le \alpha}$ is finite

2) $\bigcup_{s \in \mathbb{R}^1} \bar{B}(W_i(s), r) = (W_i(s))_{\mathbb{R}^1} + \bar{B}_r$.

Next, prove : $(W_i(s))_{\mathbb{R}^1}$ is closed.

$\forall (X_n) \subset (W_i(s))_{\mathbb{R}^1}$. (X_n) will be bdd.

Since $W_i(s)$ is transient $\Rightarrow \exists [a,b]$. s.t.

$(X_n) \subset (W_i(s))_{[a,b]}$.

Note $W_i(s)$ is conti. $\underset{cpt}{\overset{[a,b]}{\Rightarrow}}$ $(W_i(s))_{[a,b]}$ cpt.

if $X_n \to X$. then $X \in (W_i(s))_{\mathbb{R}^1}$.

Thm (Connecting property)

For $d \geq 3$. $\alpha, r > 0$. Set $G = (V, E)$. defined by each vertex in V is trajectory in W. (Y_1, Y_2) is a edge in E if $Y_1 \cap Y_2 \neq \emptyset$. in $\mathbb{R}^d$.

$\Rightarrow \mathbb{P}(\operatorname{diam} G = [\frac{d-2}{2}]) = 1$

Rmk: It means any two sausages can be connected via no more $[\frac{d-2}{2}] - 1$ sausages.

Cor. $\forall \alpha, r > 0$. $\mathcal{I}_r^\alpha$ is a.s. connected.

(3) Occupation Time :

① Def. For $A \subset \mathcal{B}_{\mathbb{R}^d}$. $\mathcal{L}_\alpha(w)(A) =: \sum_{\alpha_i \leq \alpha} \int_{\mathbb{R}^1} I_{\{W_i(s) \in A\}} ds$

$= \langle w, \int_{\mathbb{R}} I_A(w(s)) ds \otimes I_{[0,\alpha]} \rangle$. occupation time of $w \in \Omega$. in A.

prop. $\mathcal{L}_\alpha$ support on $\mathcal{L}_0^+$. and $\mathbb{E}(\mathcal{L}_\alpha(w)(A)) = \alpha|A|$.

Pf: $LHS \overset{\text{chapbell}}{=} \alpha \langle V, \int_{\mathbb{R}} I_A(w(s))ds \rangle$

$$\underset{V=\mathbb{1}^* e_B}{\overset{B \supset A}{=}} \alpha \, \mathbb{E}_{e_B}\left(\int_0^\infty I_A(X_s)ds \right)$$

$$= \alpha \int e_B(dy) \, G(y,y') I_A(y') dy' = \alpha |A|.$$

prop. If V is bdd. measurable. cpt support on $\mathbb{R}^d$.

$\| G|V| \|_{L^\infty(\mathbb{R}^d)} < 1$. Then: $I - GV$ is bdd

invertible on $L^\infty(\mathbb{R}^d)$. and $\forall \alpha > 0$. we have:

$$\mathbb{E}\left(e^{\langle \mathcal{L}_\alpha, V \rangle} \right) = e^{\alpha \langle V, (I-GV)^{-1} \mathbb{1} \rangle}.$$

cor. $\mathbb{E}(\langle \mathcal{L}_\alpha(w), V \rangle) = \int_{\mathbb{R}^d} \alpha V \, dy$

prop. $\mathcal{L}_\alpha \overset{\mathbb{P}}{\sim} \lambda^{-} h_\lambda \circ \mathcal{L}_{\lambda^{d-2}\alpha}$. $\lambda > 0$. where $h_\lambda(y) = \lambda y$.

Pf: It follows from the prop. of ν.

③ Scaling limit:

Denote: i) $\mathcal{L}_N =: \frac{1}{dN^2} \sum_{x \in \mathbb{Z}^d} L_{x,u_N} \delta_{x/N}$. $N > 1$. where

$(u_N) \subset \mathbb{R}^+$. positive level. L_{x,u_N} is the

occupation time at x of RW interlacement at level u_N. $L_u =:$ ~~the~~ field.

ii) $\hat{\mathcal{L}}_N =: \sqrt{\frac{d}{2N^{d-2} u_N}} \cdot (\mathcal{L}_N - \mathbb{E}(\mathcal{L}_N))$

Thm (Laplace Transf. of $L_{x,u}$)

$$\mathbb{E}\left(e^{\langle V, L_u\rangle}\right) = e^{u\langle V, (I-GV)^{-1}\mathbb{1}\rangle}.$$

where $V: \mathbb{Z}^d \to \mathbb{R}$, cpt supp. $\|G|V|\|_\infty < 1$.

Cor. $\mathbb{E}(L_{x,u}) = u$. $\forall x \in \mathbb{Z}^d$.

Pf: By differentiation

Cor. $\mathbb{E}(\mathcal{L}^N) = \frac{u_N}{N^{d-2}} \sum_{\mathbb{L}_N} \delta_y$.

Thm. (Const. intensity regime)

For $u_N = d\alpha N^{2-d}$, $\alpha > 0$. $\mathcal{L}_N \xrightarrow[N\to\infty]{d} \mathcal{L}_\alpha$.

Lemma. For $V \in C_c(\mathbb{R}^d)$. $\operatorname{supp}(V) = C(M) =: [-M, M]^d$.

i) $\sup_N \|G_N|V|\|_{L^\infty(\mathbb{L}_N)} \leq C_0(M)\|V\|_{L^\infty(C(M)\cap\mathbb{L}_N)}$

ii) $\lim_{N\to\infty} \langle V, (G_NV)^{n-1}\mathbb{1}\rangle_{\mathbb{L}_N} = \langle V, (GV)^{n-1}\mathbb{1}\rangle$.

where $G_Nf(x) =: \sum_{y\in\mathbb{L}_N} g_N(x,y)f(y)/N^d$.

$\langle f, h\rangle_{\mathbb{L}_N} =: \sum_{y\in\mathbb{L}_N} f(y)h(y)/N^d$.

Pf: i) By asymptotic estimate of g.

ii) prove: for $\forall\, W \in C_B(\mathbb{R}^d)$.

$$\lim_{N\to\infty} \sup_{C(M)\cap\mathbb{L}_N} |G_NVW(y) - GVW(y)| = 0.$$

Then induction on n

The claim follows from Riemann Sum approxi.

<u>Pf:</u> By the Laplace transf. of $L_{x,u}$.

$$\Rightarrow \mathbb{E}(e^{z\langle \mathcal{L}_N, V\rangle}) = e^{u\langle zV, (I-zG_N V)^{-1}\mathbb{1}\rangle_{L_N}}$$

$$= e^{u\sum_{n\geq 1} z^n \langle V, (G_N V)^{n-1}\mathbb{1}\rangle_{L_N}}$$

So: $\sup_N \mathbb{E}(\cosh(r\langle \mathcal{L}_N, V\rangle)) < \infty$. $r < C_0^{-1}(M)$

$\Rightarrow \langle \mathcal{L}^N, V\rangle$ is tight. $e^{z\langle \mathcal{L}_N, V\rangle}$ is u.i.

So: $\exists\, \mathcal{U}$. r.v. $\langle \mathcal{L}^{N_k}, V\rangle \xrightarrow{d} \mathcal{U}$. $\exists (N_k) \to \infty$

$$\mathbb{E}(e^{z\mathcal{U}}) = \lim_{k\to\infty} \mathbb{E}(e^{z\langle \mathcal{L}^{N_k}, V\rangle})$$

$$\overset{\text{exchange limit}}{=} e^{u\sum z^n \langle V, (GV)^{n-1}\mathbb{1}\rangle} = \mathbb{E}(e^{z\langle \mathcal{L}_u, V\rangle})$$

$$\Rightarrow \langle \mathcal{L}^N, V\rangle \xrightarrow{d} \langle \mathcal{L}_u, V\rangle. \quad \forall V \in C_c(\mathbb{R}^d)$$

<u>Thm.</u> (Low intensity regime)

For $N^{d-2}u_N \xrightarrow{N\to\infty} 0$. Then: $\mathcal{L}^N \xrightarrow{d} 0$.

<u>Thm.</u> (High intensity regime)

For $N^{d-2}u_N \xrightarrow{N\to\infty} \infty$. $V \in C_c(\mathbb{R}^d)$. Then:

$\langle \tilde{\mathcal{L}}_N, V\rangle \xrightarrow{d} N(0, \mathcal{E}(V,V))$. where $\mathcal{E}(V,V)$

$=: \int V(y)V(y')G(y-y')\,dy\,dy'$.

Pf: Set $\|V\|_\infty \le 1$. $r_N =: (2N^{d-2} u_N/d)^{\frac{1}{2}}$.

$$\mathbb{E}(e^{z\langle \hat{\mathcal{L}}_N, V\rangle}) = \mathbb{E}(e^{\frac{z}{r_N}\langle \mathcal{L}_N, V\rangle - \frac{u_N}{d}N^{d-2}\langle V, 1\rangle_{\mathbb{L}_N}})$$

$$= e^{\frac{u_N}{d}N^{d-2}\sum_{n\ge 2}\frac{z^n}{r_N^n}\langle V, (h_N V)^{n-1}1\rangle_{\mathbb{L}_N}}$$

$$\xrightarrow{r\to\infty} e^{\frac{z^2}{2}\langle V, hV\rangle}. \quad \forall |z| \le c_0^{-1}$$

Rmk: Note $S(\mathbb{R}^d)$ is dominated by $(1+|y|)^{-d-1}$.

and $\mathbb{E}(\sum_{\mathbb{L}_N}(1+|y|)^{-d-1}\mathcal{L}_{N_y, u_N}) < \infty$

So: $\hat{\mathcal{L}}_N \in S^*(\mathbb{R}^d)$ - r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$.

Cor. $\hat{\mathcal{L}}_N \xrightarrow{d} \mathbb{P}^G(\cdot)$. law of centered GFF.

with variance $E(V, V)$. on $S^*(\mathbb{R}^d)$

Pf: prove: $\forall V \in S(\mathbb{R}^d)$. $\langle \mathcal{L}^N, V\rangle \xrightarrow{d} \langle \phi, V\rangle$.

where $\phi \sim N(0, E(V, V))$.

Set $V_L =: V\chi_L \in C_0^\infty(\mathbb{R}^d)$.

Show $\lim_{L\to\infty}\sup_{N\ge 1}\mathbb{E}(\langle \hat{\mathcal{L}}_N, V - V_L\rangle^2) = 0$

for exchanging limit.

Thm. For $B \subseteq \mathbb{R}^d$. closed ball. $N^{-d}\sum_{x\in B\cap\mathbb{Z}^d}\mathcal{L}_{x,u}\delta_{x/N} =: e_N$

$\xrightarrow[N\to\infty]{v} u\,\mathcal{L}1_B$. $\mathcal{L}$ is Lebesgue measure

Pf: By ergodic Thm.