Rough Integrals

(1) <u>Motivation:</u>

For $V = \mathbb{R}^d$, $F \in C_b^2(\mathbb{R}^d ; \mathbb{R}^{m \times d})$ (regular enough)

$X \in C^\alpha$, for some $\alpha \geq 0$. $\mathcal{P} := [t_i]_1^n$ a partition

of $[s,t]$. We next define $\int_s^t F(X_r) dX_r$:

First, we assume $\alpha > \frac{1}{2}$.

By Taylor: $F(X_r) = F(X_{t_i}) + DF(X_{t_i}) X_{t_i,r} + R_2(X_{t_i}, X_r)$

where $|R_2(X_{t_i}, X_r)| \lesssim |X_{t_i,r}|^2$.

So $\int_s^t F(X_r) dX_r = \sum_0^{n-1} \int_{t_i}^{t_{i+1}} \boxed{}$

$\overset{\text{AS}}{=} \sum_0^{n-1} \left[F(X_{t_i}) X_{t_i, t_{i+1}} + DF(X_{t_i}) \int_{t_i}^{t_{i+1}} X_{t_i,r} \, dX_r + \int_{t_i}^{t_{i+1}} R_2 \right]$

$\overset{\Delta}{=} A_n + B_n + C_n$.

Note $\left| \int_{t_i}^{t_{i+1}} X_{t_i,r} \, dX_r \right| \lesssim |t_{i+1} - t_i|^{2\alpha}$

$\left| \int_{t_i}^{t_{i+1}} R_2(X_{t_i}, X_r) \, dX_r \right| \lesssim |t_{i+1} - t_i|^{3\alpha}$

choose $\{t_i\} = \mathcal{P}_n = \{ s + i(t-s)/2^n \}_i$

$\Rightarrow$ $\lim_{n\to\infty} |B_n| \lesssim |DF|_\infty \lim_n 2^n \cdot ((t-s)/2^n)^{2\alpha} = 0$

$\lim_{n\to\infty} |C_n| \lesssim \lim_n 2^n \cdot ((t-s)/2^n)^{3\alpha} = 0$

So it reduces to the case of RS integral. And when we consider $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. We see: $C_n \to 0$ still but not for B_n.

We hope $\int_s^t F(X_r)\, dX_r = \lim_{|\rho| \to 0} \sum_{[t_i, t_{i+1}] \in \rho} (F(X_{t_i}) X_{t_i, t_{i+1}} +$

$$DF(X_{t_i}) \int_{t_i}^{t_{i+1}} X_{t_i, r} \otimes dX_r)$$

But here $\int_{t_i}^{t_{i+1}} X_{t_i, r} \otimes dX_r$ won't be defined in RS integral sense. And we want to postulate it as a abstract process $\mathbb{X}_{s,t}$ with some algebra/regular cond. to let limit converge.

Lem. $F \in C_B^2(\mathbb{R}^d; \mathbb{R}^{m \times d})$. $(X, \mathbb{X}) \in \mathscr{C}^\alpha$. $\alpha \in (\frac{1}{3}, \frac{1}{2}]$.

For $Y_t := F(X_t)$, $Y_t' := DF(X_t)$, $R^Y_{s,t} := Y_{s,t} - Y_s' X_{s,t}$

$\Rightarrow Y \in C^\alpha(I, \mathbb{R}^{m \times d})$, $Y_t' \in C^\alpha(I, L(\mathbb{R}^d, \mathbb{R}^{m \times d}))$

$R^Y \in C_2^{2\alpha}(I, \mathbb{R}^{m \times d})$. Besides, $\|Y\|_\alpha \le \|DF\|_\infty \|X\|_\alpha$.

$\|Y'\|_\alpha \le \|D^2 F\|_\infty \|X\|_\alpha$. $\|R^Y\|_{2\alpha} \le \frac{1}{2} \|D^2 F\|_\infty \|X\|_\alpha^2$.

Pf: Parts of Y, Y' are trivial.

And we see $R^Y_{s,t} = \frac{1}{2} D^2 F(X_s + \xi X_{s,t}) \cdot (X_{s,t} \otimes X_{s,t})$ for some $\xi \in (0,1)$ by intermediate value Thm.

(2) <u>Sewing Lem.</u>

Motivation: Note from (1). No matter $X \in C^\alpha$. $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ or $\alpha > \frac{1}{2}$. Its integral approxi. may have form: $\sum_{2i \in \mathcal{P}} \xi_{2i, 2i+1} \xrightarrow{|\mathcal{P}| \to 0} \int \square$. $\xi : I^2 \to \mathbb{R}^m$.

Recall in (1). We ~~set~~ $\xi_{s,t} = F(X_s) X_{s,t}$ and $\xi_{s,t} = F(X_s) X_{s,t} + DF(X_s) \mathbb{X}_{s,t}$ which yields good approxi.: $\left| \int_s^t F(X_r) dX_r - \xi_{s,t} \right| \lesssim |t-s|^{3\alpha}$ or $|t-s|^{2\alpha}$ for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ or $\alpha > \frac{1}{2}$

i.e. $\xi_{s,t}$ locally approxi. $\int_s^t F(X_r) dX_r$ will be better than linear.

<u>Rmk</u>: Note that germ ξ isn't additive in both cases. But $\int_s^t F(X_r) dX_r$ is additive. We call this step of patching together the non-additive germ ξ to a additive limit integral map by sewing.

<u>Def</u>: For V, W Banach space with Frechét diff.

$C_2^{\alpha, \beta}(I, W) := \{ \xi : I^2 \to W \mid \xi_{t,t} = 0,\ \forall t \in I.$

s.t. $\|\xi\|_{\alpha,\beta} := \|\xi\|_\alpha + \|\delta\xi\|_\beta < \infty\}$.

$\delta\xi : I^3 \to W$, $\delta\xi_{s,u,t} := \xi_{s,t} - \xi_{s,u} - \xi_{u,t}$ and

$\|\delta\xi\|_\beta := \sup_{s<u<t} |\delta\xi_{s,u,t}| / |t-s|^\beta$.

Lem. (Sewing Lem.)

For $0<\alpha\le 1<\beta$. Then: $\exists$ unique BLO $\mathcal{I} : C_2^{\alpha,\beta}(I,W) \to C^\alpha(I,W)$ s.t. $(\mathcal{I}\xi)_0 = 0$ and $\exists C = C(\beta, \|\delta\xi\|_\beta) > 0$. $|(\mathcal{I}\xi)_{s,t} - \xi_{s,t}| \le C|t-s|^\beta$.

Rmk: $(\mathcal{I}\xi)_0 = 0$ is for uniqueness.

Pf: 1) Uniqueness: For two process $I, \bar{I}$ s.t. satisfy cond. $\Rightarrow |(I-\bar{I})_{s,t}| \lesssim |t-s|^\beta$ for $\beta>1$. So: $I \equiv \bar{I}$

And we can see the only candidate is $(\mathcal{I}\xi)_{s,t} = \lim_{|\mathcal{P}|\to 0} \sum \xi_{z_i, z_{i+1}}$

since for $\mathcal{P}_n := \{z_k\} = \{s + \frac{(t-s)k}{2^n}\}$.

$|(\mathcal{I}\xi)_{s,t} - \sum \xi_{z_i, z_{i+1}}| =$

$|\sum_{\mathcal{P}_n} ((\mathcal{I}\xi)_{z_i,z_{i+1}} - \xi_{z_i,z_{i+1}})| \lesssim 2^n \cdot 2^{-n\beta} \to 0$.

Besides, we also show $\mathcal{I}$ is linear.

2) Set $I^0_{s,t} \triangleq \xi_{s,t}$ and iteratively def:

$I^{n+1}_{s,t} \overset{\Delta}{=} \sum_{\mathcal{P}_{n+1}} \xi_{u,v} = I^{n}_{s,t} - \sum_{\mathcal{P}_n} \delta\xi_{u,m,v}$

$\Rightarrow |I^{n+1}_{s,t} - I^{n}_{s,t}| \leq 2^{n} \cdot (2^{-n}|t-s|)^{\beta} \|\delta\xi\|_{\beta}$.

So $\sum_n |I^{n+1}_{s,t} - I^{n}_{s,t}| < \infty \Rightarrow (I^{n}_{s,t})$ is Cauchy

Denote its limit by $I_{s,t}$.

$\Rightarrow |I_{s,t} - \xi_{s,t}| \leq \sum_{n\geq 0} |I^{n+1}_{s,t} - I^{n}_{s,t}| \lesssim \|\delta\xi\|_{\beta} |t-s|^{\beta}$

Also $|I_{s,t}| \leq |\xi_{s,t}| + \|\delta\xi\|_{\beta} |t-s|^{\beta} \Rightarrow I \in C^{\alpha}$

So $I_{s,t} =: (\mathcal{I}\xi)_{s,t}$ is what we need.

And $\mathcal{I}: \xi \in C_2^{\alpha,\beta} \mapsto \mathcal{I}\xi \in C^{\alpha}$ is BLO.

3) For additivity of I, we note that $I_{s,t} = I_{s,u} + I_{u,t}$ for $\forall\, [s,t] = [\frac{\ell}{2^n}, \frac{\ell+1}{2^n}]$ and $u = \frac{s+t}{2}$ from $I^{n+1}_{s,t} = I^{n}_{s,u} + I^{n}_{u,t}$ by set $n \to \infty$ on both sides

Hence $I_{k/2^m,\, \ell/2^m} = \sum_{k}^{\ell-1} I_{i/2^m,\, (i+1)/2^m}$.

And we can approxi. $[s,t]$ by $2^{-k}[\ell, m]$.

4) It can be extended to any partition s.t. $|\mathcal{P}| \to 0$.

<u>prop.</u> $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^{\alpha}(I, V)$, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $F \in C_b^2(V, L(V,W))$. Then rough integral $\int_s^t F(X_r)\, d\mathbf{X}_r$

exists for $\forall s,t \in I$. defined by

$\lim_{|\mathcal{P}|\to 0} \sum_{\mathcal{P}} (F(X_{t_i}) X_{t_i,t_{i+1}} + DF(X_{t_i}) \mathbb{X}_{t_i,t_{i+1}})$ satisfy

$$\left| \int_s^t F(X_r)\,d\mathbf{X}_r - F(X_s)X_{s,t} - DF(X_s)\mathbb{X}_{s,t} \right|$$

$$\le C(\alpha)\, \|F\|_{C_b^3} (\|X\|_\alpha^3 + \|X\|_\alpha \|\mathbb{X}\|_{2\alpha})\, |t-s|^{3\alpha}.$$

Besides. $t \mapsto \int_0^t F(X_r)\,d\mathbf{X}_r \in C^\alpha$ and satisfies

$$\left\| \int_0^\cdot F(X_r)\,d\mathbf{X}_r \right\|_\alpha \le C \|F\|_{C_b^2} (|||\mathbf{X}|||_\alpha \vee |||\mathbf{X}|||_\alpha^{1/\alpha})$$

<u>Rmk</u>: If try to consider $\Xi_{s,t} = F(X_s) X_{s,t}$.

We see $\delta\Xi \in C^\beta$ only for $\beta < 2\alpha \le 1$.

won't satisfy cond. of sewing lem.

<u>Pf</u>: Set $Y_t = F(X_t)$. $\Xi_{s,t} = Y_s X_{s,t} + Y_s' \mathbb{X}_{s,t} \in C^\alpha$.

Using Chen's relation, we see:

$$\delta\Xi_{s,u,t} = -R^Y_{s,u} X_{u,t} - Y'_{s,u} \mathbb{X}_{u,t} \overset{\text{Lem}}{\underset{(1)}{\in}} C^{3\alpha}.$$

So: We can apply sewing Lemma.

Combine with estimate:

$$\|\delta\Xi\|_{3\alpha} \le \frac{1}{2} \|D^2 F\|_\infty \|X\|_\alpha^3 + \|D^2 F\|_\infty \|X\|_\alpha \|\mathbb{X}\|_{2\alpha}.$$

(3) <u>Integration on Controlled RPs</u>:

We want to integrate on a larger class. The key is to use $\delta\Xi_{s,u,t} = -R^Y_{s,u} X_{u,t} - Y'_{s,u} \mathbb{X}_{u,t}$ with their regularity ($\Rightarrow$ extend out of one-form)

Def: $\bar{W} := L(V, W)$. $\alpha \in (0, \frac{1}{2}]$. $X \in C^\alpha(I, V)$. $Y \in C^\alpha(I, \bar{W})$ is controlled RP by X if $\exists\, Y' \in C^\alpha(I, L(V, \bar{W}))$. s.t. $R^Y_{s,t} = Y_{s,t} - Y'_s X_{s,t} \in C_2^{2\alpha}(I, \bar{W})$

Denote space of such pairs (Y, Y') by $\mathcal{D}_X^{2\alpha}(I, \bar{W})$.

Rmk: i) Note no true RP in the def but only C^α-path.

ii) Y' isn't uniquely determined by X. Y sometimes. It depends on the intense of roughness of X. Y.

iii) Elements in $\mathcal{D}_X^{2\alpha}$ is path looking like X in small scale $Y_{s,t} \approx Y'_s X_{s,t}$

iv) Fix X. $\mathcal{D}_X^{2\alpha}$ is LS with seminorm $\|Y, Y'\|_{X,2\alpha} := \|Y'\|_\alpha + \|R^Y\|_{2\alpha}$ & norm $\|Y, Y'\|_{\mathcal{D}_X^{2\alpha}} := |Y_0| + |Y'_0| + \|Y, Y'\|_{X,2\alpha}$.

Lem. For $(Y, Y') \in \mathcal{D}_X^{2\alpha} \Rightarrow \|Y\|_\alpha \leq C \|Y, Y'\|_{\mathcal{D}_X^{2\alpha}}$

Pf: By expression: $Y_{s,t} = Y'_s X_{s,t} + R^Y_{s,t}$

Rmk: That's why we don't need the term $\|Y\|_\alpha$ in def of $\|\cdot\|_{\mathcal{D}_X^{2\alpha}}$.

Cor. $(D_X^{2\gamma}, \|\cdot\|_{D_X^{2\gamma}})$ is Banach space.

Thm. For $\gamma \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\gamma(I, V)$. (Y, Y') $\in D_X^{2\gamma}(I, L(V, W))$. Then:

$$\int_s^t Y_r \, d\mathbf{X}_r := \lim_{|\mathcal{P}| \to 0} \sum (\underbrace{Y_{t_i} X_{t_i, t_{i+1}} + Y'_{t_i} \mathbb{X}_{t_i, t_{i+1}}}_{\triangleq \, \Xi_{s,t}})$$

exists, $t \mapsto \int_0^t Y_r \, d\mathbf{X}_r \in C^\gamma$. With estimate:

$$\left| \int_s^t Y_r \, d\mathbf{X}_r - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t} \right|$$

$$\leq C_\gamma (\|X\|_\gamma \|R^Y\|_{2\gamma} + \|\mathbb{X}\|_{2\gamma} \|Y'\|_\gamma) |t-s|^{3\gamma}$$

Besides, $(\int_0^\cdot Y_r \, d\mathbf{X}_r, Y) \in D_X^{2\gamma}$. $(Y, Y') \in D_X^{2\gamma}(I, L(V, W)) \mapsto (\int_0^\cdot Y_r \, d\mathbf{X}_r, Y) \in D_X^{2\gamma}(I, W)$ is BLO.

Rmk: $\int Y_r \, d\mathbf{X}_r$ will depend on $Y, Y', X, \mathbb{X}$. But Y' is invisible

Pf: i) Existence totally follow from before since $\|\delta \Xi\|_{3\gamma} \leq \|X\|_\gamma \|R^Y\|_{2\gamma} + \|\mathbb{X}\|_{2\gamma} \|Y'\|_\gamma$.

We also obtain the estimate

2) From the estimate: $\int_s^t Y_s \, d\mathbf{X}_r - Y_s X_{s,t}$ $= Y'_s \mathbb{X}_{s,t} + C|t-s|^{2\gamma} \in C^{2\gamma}$.

So: $(\int_0^\cdot Y_s \, d\mathbf{X}_s, Y) \in D_X^{2\gamma}(I, W)$

As for continuity of the map:

$$\|\textstyle\int Y_s\,d\mathbf{X}, Y\|_{D_X^{2\alpha}} \lesssim |Y_0| + \|Y\|_\alpha + \|Y'\|_\infty \|\mathbb{X}\|_{2\alpha} +$$

$$C T^{\alpha}\left(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|Y'\|_\alpha\right)\Big) \lesssim \|Y, Y'\|_{D_X^{2\alpha}}$$

<u>e.g.</u> $f \in C^{2\alpha}(I, V\otimes V)$, $\mathbf{X}, \widetilde{\mathbf{X}} \in \mathscr{C}^{\alpha}(I, V)$. s.t. $X = \widetilde{X}$ and $\widetilde{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + f_{s,t}$. let $(Y, Y') \in D_X^{2\alpha}(I, L(V,W))$

$\Rightarrow \int_s^t Y_r\,d\widetilde{\mathbf{X}}_r = \int_s^t Y_r\,d\mathbf{X}_r + \int_s^t Y_r'\,df_r$ follows from construction.

<u>Lem.</u> For $X, Y \in C^1$. $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{L}(C^1)$. Then:

$\int Y_r\,d\mathbf{X}_r = \int Y_r\,dX_r$ in sense of RS integral

<u>Pf</u>: germ of LHS is $Y_s X_{s,t} + (DY)_s \int_s^t X_{s,r}\otimes dX_r$

germ of RHS is $Y_s X_{s,t}$ denoted by $\bar{\mathfrak{z}}, \mathfrak{z}$.

$\Rightarrow \bar{\mathfrak{z}} - \mathfrak{z} \in C_2^{\beta}$, for $\beta > 1$.

So $|(\mathcal{J}\bar{\mathfrak{z}})_t - (\mathcal{J}\mathfrak{z})_t| \le \lim_{|\mathcal{P}|\to 0} \sum_{\mathcal{P}} |\bar{\mathfrak{z}}_{t_i,t_{i+1}} - \mathfrak{z}_{t_i,t_{i+1}}|$

$\le C \lim 2^n \cdot 2^{-n\beta} = 0$. $\mathcal{P} = [\,iT/2^n\,]$.

<u>Rmk</u>: In this case: the rough integral is indept of Y' and $\mathbb{X}$.

(4) <u>Stability</u>:

Recall for RS integral, $(Y, X) \in C^1 \times C^1 \mapsto \int Y\,dX$

$\in C([0,1], \|\cdot\|_\infty)$ isn't cont.

Next, we want to study regularity of $(Y, \mathbf{X}) \in \mathcal{D}_X^{2\alpha} \times \mathcal{C}^\alpha \mapsto \int Y_r \, d\mathbf{X}_r \in \mathcal{D}_X^{2\alpha}$

For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{X}, \bar{\mathbf{X}} \in \mathcal{C}^\alpha(I, V)$. $(Y, Y') \in \mathcal{D}_X^{2\alpha}(I, L(V, W))$. $(\bar{Y}, \bar{Y}') \in \mathcal{D}_{\bar{X}}^{2\alpha}(I, L(V, W))$.

Since (Y, Y'), $(\bar{Y}, \bar{Y}')$ live in different Banach space. "distance" between them doesn't make sense. We can still denote:

$\| Y, Y'; \bar{Y}, \bar{Y}' \|_{X, \bar{X}, 2\alpha} := \| Y' - \bar{Y}' \|_\alpha + \| R^Y - R^{\bar{Y}} \|_{2\alpha}$

<u>Rmk</u>: It's not a true metric even for $X = \bar{X}$. Since $\{(Y + cX + \tilde{c}, Y' + c)\}_c$ has 0 distance ($\|\cdot\|_\alpha$, $\|\cdot\|_{2\alpha}$ are both semi-norms)

Set $Z = \int Y \, d\mathbf{X}$ and $\bar{Z} = \int \bar{Y} \, d\bar{\mathbf{X}}$.

<u>Prop.</u> For $M > 0$, s.t. $(|Y_0'| + \| Y, Y' \|_{X, 2\alpha}) \vee (\| X \|_\alpha + \| \mathbb{X} \|_{2\alpha}) \le M$. Assume it also works for $(\bar{Y}, \bar{\mathbf{X}})$.

Then: $\| Z - \bar{Z} \|_\alpha \vee \| Z, Z'; \bar{Z}, \bar{Z}' \|_{X, \bar{X}, 2\alpha} \le$

$$C(\alpha, M, T) \, (\rho_\alpha(\mathbf{X}, \bar{\mathbf{X}}) + |Y_0' - \bar{Y}_0'| + |Y_0 - \bar{Y}_0| + T^\alpha \| Y, Y'; \bar{Y}, \bar{Y}' \|_{X, \bar{X}, 2\alpha})$$

Rmk: Note the const. C depends on μ. (i.e. rely on (Y, Ξ)). So for uniform estimate, we need to get bdd family and then we have truly conti.

(5) True roughness:

Next, we want to determine when Y' is uniquely determined by X and Y?

Rmk: Recall if $X, Y \in C^{2r}$. $\Rightarrow \forall Y \in C$ works.

Note if set $V = W = \mathbb{R}'$. let $\frac{|X_{t,t_n}|}{|t_n - t|^{2r}} \xrightarrow{t_n \downarrow t} \infty$

Since $Y_t' = \frac{Y_{t,t_n}}{X_{t,t_n}} - \frac{R^Y_{t,t_n}}{|t-t_n|^{2r}} \cdot \frac{|t_n - t|^{2r}}{X_{t,t_n}}$

We set $Y_t' = \lim_{t_n \downarrow t} Y_{t,t_n} / X_{t,t_n}$ exists uniquely.

Def: $X \in C^r(I, V)$ is rough at time t if

$\forall v^* \in V^* / \{0\}$. $\overline{\lim_{z \downarrow t}} |v^*(X_{t,z})| / |z-t|^{2r} = \infty$

X is truly rough if it's rough at $t \in D \subset I$. dense set

Rmk: Since Y_t' conti. So it's determined by D.

prop. If X is rough at time $t \in I$. Then $\forall$

$(Y, Y') \in \mathcal{D}_X^{2\alpha}(I, W)$. s.t. $\varlimsup_{z \downarrow t} |Y_{t,z}| / |z-t|^{2\alpha} < \infty$. $\Rightarrow Y_t' = 0$.

<u>Cor.</u> If X is truly rough. (Y, Y'), $(Y, \tilde{Y}') \in \mathcal{D}_X^{2\alpha}$. Then: $Y' = \tilde{Y}'$.

<u>Pf:</u> Note $(Y - Y, \tilde{Y}' - Y') \in \mathcal{D}_X^{2\alpha}$

<u>Pf:</u> Note $\frac{Y_t' X_{t,z}}{|z-t|^{2\alpha}} = \frac{Y_{t,z}}{|z-t|^{2\alpha}} - \frac{R^Y_{s,t}}{|z-t|^{2\alpha}}$

its RHS is bdd as $z \downarrow t$ by cond.

$\forall w^* \in W^*$. Set $v^* = w^* \circ Y_t'$

We see $\varlimsup_{z \downarrow t} |v^*(X_{t,z})| / |z-t|^{2\alpha} < \infty$.

$\Rightarrow v^* \equiv 0 \Rightarrow Y_t' = 0$.

<u>Prop.</u> (Doob Meyer's for rough paths)

$\mathbf{X} \in \mathcal{C}^\alpha$. X is truly rough. (Y, Y'), $(\tilde{Y}, \tilde{Y}') \in \mathcal{D}_X^{2\alpha}(I, L(V, W))$ and $Z, \tilde{Z} \in C(I, W)$.

Then: $\int_0^t Y_s \, d\mathbf{X}_s + \int_0^t Z_s \, ds = \int_0^t \tilde{Y}_s \, d\mathbf{X}_s + \int \tilde{Z}_s \, ds$

for $\forall t \in I \Rightarrow (Y, Y') = (\tilde{Y}, \tilde{Y}')$. $Z = \tilde{Z}$.

<u>Rmk:</u> It's analogous to Doob-Meyer decomp.

for semimart. : $M \in \mathcal{M}^c_{loc}$. $Y, \tilde{Y}, Z, \tilde{Z} \in C(I)$

$$\int_0^t Y_s \, dM_s + \int_0^t Z_s \, ds = \int_0^t \tilde{Y}_s \, dM_s + \int_0^t \tilde{Z}_s \, ds$$

$\forall t \in I \Rightarrow Y = \tilde{Y}, \; Z = \tilde{Z}$ a.s.

<u>Pf</u>: Note $\int_0^{\cdot} Z_s - \tilde{Z}_s \, ds \in C^1(I)$. So:

$$\overline{\lim_{r \downarrow t}} \int_t^r Y_s - \tilde{Y}_s \, dX_s \Big/ |r-t|^{2\gamma} =$$

$$\overline{\lim_{r \downarrow t}} \int_t^r Z_s - \tilde{Z}_s \Big/ \square < \infty. \text{ for } t \in D \text{ dense.}$$

Conti'

$\Rightarrow Y_s = \tilde{Y}_s$ on $I \Rightarrow Y_s' = \tilde{Y}_s'$ on I

$\Rightarrow \int_0^{\cdot} (Y, Y') \, d\mathbf{X} = \int_0^{\cdot} (\tilde{Y}, \tilde{Y}') \, d\mathbf{X}$ on I

$\Rightarrow \int_0^{\cdot} Z_s \, ds = \int_0^{\cdot} \tilde{Z}_s \, ds \Rightarrow Z_s = \tilde{Z}_s$.

<u>Prop</u>. $V = \mathbb{R}^d$. B is d-dim SBM. Then we have:

$P(B$ is truly rough w.r.t $\alpha \in [\frac{1}{4}, \frac{1}{2})) = 1$.

<u>Pf</u>: By law of iterated logarithm.