Comparison with Sto-Ana.

(1) Integrations:

For $U = \mathbb{R}^d$. $B = (B^1 \cdots B^d)$, d-dim SBM on $I = [0, T]$. on complete filtered prob. space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{[0,T]}, \mathbb{P})$. $(\mathcal{F}_t)$ is right-conti.

Next, denote $\{N_i\}_i$ are $\mathbb{P}$-null sets $\in \mathcal{F}_0$.

Prop. For $\alpha \in (\frac{1}{3}, \frac{1}{2})$, $\mathbf{B}^I(\omega) = (B(\omega), \mathbb{B}^I(\omega)) \in \mathcal{C}^\alpha(I, \mathbb{R}^d)$ for $\omega \in N_1^c$. Assume $(Y(\omega), Y'(\omega)) \in D^{2\alpha}_{B(\omega)}(I, \mathbb{R}^{m \times d})$ for $\omega \in N_2^c$ and Y is $\mathcal{F}_t$-adapted

Then: $\mathbb{P}(\int_0^t Y_r(\omega)\, d\mathbf{B}^I(\omega) = (\int_0^t Y_r\, dB_r)(\omega)$ on $[0, T]) = 1$.

Pf: Note LHS & RHS are both t-conti.

So we can just fix $t \in [0, T]$.

(WLOG set $T = 1$. $t = 1$)

Note that $\int_0^1 Y_t\, d\mathbf{B}^I_t = \lim \sum Y_{t_i} B_{t_i, t_{i+1}} + Y'_{t_i} \mathbb{B}^I_{t_i, t_{i+1}}$ for $\omega \in N_1^c \cap N_2^c$.

$\int_0^1 Y_t\, dB_t = \lim \sum Y_{t_i} B_{t_i, t_{i+1}}$ $\omega \in N_3^c$.

Next, we prove: $\sum Y'_{z_i} \mathbb{B}^{\mathbb{I}}_{z_i, z_{i+1}} \to 0$ in L^2.

By localization: assume $\sup_{\bigcap_{i}^{3} N_i^c} |Y'_t(\omega)|_\infty \le M$. $\forall t$

Then: $\|LHS\|_{L^2}^2 \le \sum \|Y'_{z_i} \mathbb{B}^{\mathbb{I}}_{z_i, z_{i+1}}\|^2_{L^2} \le M^2 \sum \|\mathbb{B}^{\mathbb{I}}_{z_i, z_{i+1}}\|^2_{L^2}$

$$\sim M^2 \sum |z_i - z_{i+1}|^2 \le T M^2 |\mathcal{P}| \to 0$$

<u>Prop.</u> Under condition above and in addition with $\langle Y, B\rangle$ exists on $\mathcal{I}$. Then:

$$\mathbb{P}\left(\int_0^t Y_r(\omega)\, dB^S_r(\omega) = \left(\int_0^t Y_r \circ dB_r\right)(\omega) \text{ on } [0,T]\right) = 1$$

<u>Pf:</u> RHS $= \int_0^t Y_r\, dB_r + \frac{1}{2} \langle Y, B\rangle_t$

LHS $= \int_0^t Y_r\, d\mathbb{B}^{\mathbb{I}}_r + \lim \sum \frac{1}{2}(z_{i+1} - z_i) Y'_{z_i}$

$$\overset{F(t)=t}{=} \int_0^t Y_r\, d\mathbb{B}^{\mathbb{I}}_r + \frac{1}{2}\int_0^t Y'_r\, dr$$

while $Y_{s,t} B_{s,t} = Y'_s B_{s,t} \otimes B_{s,t} + R^Y_{s,t} B_{s,t}$

$$= 2 Y'_s \,\mathrm{Sym}(\mathbb{B}^{\mathbb{I}}_{s,t}) + Y'_s(t-s) + \square$$

$\sum Y'_s \,\mathrm{Sym}(\mathbb{B}^{\mathbb{I}}_{s,t}) \xrightarrow{L^2} 0$ from above

$|\sum R^Y_{s,t} B_{s,t}| \le \|R^Y\|_{2\alpha} \|B\|_\alpha \sum |t-s|^{3\alpha} \to 0$.

So $\langle Y, B\rangle = \lim \sum Y_{s,t} B_{s,t} = \int Y'\, dr$

(2) <u>Rough Itô formula:</u>

Recall classical Itô formula, classical. $f \in C_b^2$:

$G(X_t) = G(X_s) + \int_s^t \nabla G(X_r) \cdot dX_r + \frac{1}{2}\int_s^t D^2 G(X_r) : d\langle X, X\rangle_r$

where integrals of RHS is defined in RS.

Rmk: $X \in C^1 \Rightarrow \langle X, X\rangle = 0$. We get chain rule.

Note $\int_s^t D^2 G(X_r) : d\langle X\rangle_r = \lim \sum D^2 G(X_{t_i}) X_{t_i, t_{i+1}} \otimes X_{t_i, t_{i+1}}$

If we assume $\mathbf{X} \in \mathscr{C}_g^\alpha$. $F = DG \in C_B^2$.

Since $\mathrm{Sym}(\mathbb{X}_{u,v}) = \frac{1}{2} X_{u,v} \otimes X_{u,v}$ and DF is symmetric so $DF \cdot \mathrm{Ant}(\mathbb{X}_{u,v}) = 0$.

$\Rightarrow G(X_t) - G(X_s) = \lim \sum (\nabla G(X_{t_i}) \cdot X_{t_i, t_{i+1}} + D^2 G(X_{t_i}) \cdot \mathbb{X}_{t_i, t_{i+1}})$

$= \int DG(X_r)\, d\mathbf{X}_r = \int F(X_r)\, d\mathbf{X}_r$.

Rmk: We don't necessarily assume $V = \mathbb{R}^d$ above since Itô's formula works for Banach space-valued func. by Taylor expansion

Lem. $\mathbf{X} \in \mathscr{C}_g^\alpha(I, V)$ for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and $G \in C_B^3(V, W)$. Then: $G(X_t) = G(X_s) + \int_s^t DG(X_r)\, d\mathbf{X}_r$.

Cor. Replace "$X \in \mathscr{C}_g^\alpha(I, V)$" by "$X \in \mathscr{C}^\alpha(I, V)$"

We have: $G(X_t) = G(X_s) + \int_s^t DG(X_r) \cdot d\mathbf{X}_r + \int_s^t D^2 G(X_r) : d(\frac{1}{2} X_r \otimes X_r - \mathrm{Sym}(\mathbb{X}_r))$

Rmk: The last integral is defined in Young sense ($\tau + 2\tau > 1$).

Note the antisym part doesn't involve in the computation above. It motivates:

Def: For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $X \in C^\alpha(I, V)$, $S \in C_2^{2\alpha}(I, \mathrm{Sym}(V \otimes V))$, (X, S) is α-reduced rough path if reduced Chen's relation hold:

$S_{s,t} - S_{s,u} - S_{u,t} = \mathrm{Sym}(X_{s,u} \otimes X_{u,t})$, $\forall s, u, t \in I$.

Denote the space of them by $\mathscr{C}_r^\alpha(I, V)$.

Rmk: i) For $\mathbb{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha \Rightarrow (X, \mathrm{Sym}(\mathbb{X})) \in \mathscr{C}_r^\alpha$. (Note take $\mathrm{sym}(\cdot)$ on both sides of Chen's relation)

ii) For $X \in C^\alpha$, it has a trivial lift $S_{s,t} = \frac{1}{2} X_{s,t} \otimes X_{s,t}$ to reduced RPs. In contrast to common RPs, its "trivial" lift (has explicit form) doesn't exist. Besides, its reduced RP lift isn't unique necessarily.

iii) There's a natural reducing operator:

$\Delta : (X, \mathbb{X}) \in \mathscr{C}^{\alpha} \mapsto (X, \mathrm{Sym}(\mathbb{X})) \in \mathscr{C}^{\alpha}_{r}$.

And for $\mathbf{X} \in \mathscr{C}^{\alpha}_{g}$:

$\Delta(\mathbf{X}) = (X, \frac{1}{2} X \otimes X)$ trivial lift.

__Lem.__ For $X \in C^{\alpha}(I, V)$. Then: $(X, \mathbb{S}) \in \mathscr{C}^{\alpha}_{r}(I, V)$

$\iff \mathbb{S} \in C^{2\alpha}_{2}(I, \mathrm{Sym}(V \otimes V))$ with $\mathbb{S}_{s,t} = \frac{1}{2} X_{s,t} \otimes X_{s,t} + Y_{s,t}$ for some $Y \in C^{2\alpha}(I, \mathrm{Sym}(V \otimes V))$

__Pf:__ Analogous as case of common RPs.

__Def:__ $\mathbf{X} = (X, \mathbb{S}) \in \mathscr{C}^{\alpha}_{r}$. Bracket of $\mathbf{X}$ is $[\mathbf{X}]$

$: t \in I \mapsto [\mathbf{X}]_t := X_{0,t} \otimes X_{0,t} - 2\mathbb{S}_{0,t} \in \mathrm{Sym}(V \otimes V)$

__Rmk:__ Note by Lem above: $\delta[X]_{s,t}$ is increment of $C^{2\alpha}(I, \mathrm{Sym}(V^{\otimes 2}))$.

__Prop.__ For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $G \in C^{3}_{B}(V, W)$. $\mathbf{X} = (X, \mathbb{S}) \in \mathscr{C}^{\alpha}_{r}(I, V)$. Then: $\forall s, t \in I$. We have $G(X_t)$

$= G(X_s) + \int_s^t \nabla G(X_r) \cdot d\mathbf{X}_r + \frac{1}{2} \int_s^t D^2 G(X_r) : d[\mathbf{X}]_r$

where $\int_s^t \nabla G(X_r) \cdot d\mathbf{X}_r$

$:= \lim \sum (\nabla G(X_{t_i}) \cdot X_{t_i, t_{i+1}} + D^2 G(X_{t_i}) \mathbb{S}_{t_i, t_{i+1}})$

Rmk: i) For $(X, \mathbb{X}) \in \mathscr{C}^\alpha$. The result can be applied on $(X, \mathrm{Sym}(\mathbb{X}))$. Note that $\int_s^t \nabla G(X_r)\, d\mathbf{X}_r$ is still full rough integral since $\mathrm{Ant}(\cdot)$ part vanish

ii) For $(X, \mathbb{X}) \in \mathscr{C}_g^\alpha$, apply it on $(X, \mathrm{Sym}(\mathbb{X})) \Rightarrow [\mathbf{X}] = 0$. $\int_0$: it's reduced to the case before.

iii) Careful that the result only works for gradient one-form.

Pf: Note by Itô's:

$$G(X_t) = G(X_s) + \lim \Sigma \big(\nabla G(X_{2^i}) X_{2^i, 2^{i+1}} + \tfrac{1}{2} D^2 G(X_{2^i}) (X_{2^i, 2^{i+1}} \otimes X_{2^i, 2^{i+1}}) \big)$$

$$= G(X_s) + \lim \Sigma \big(\nabla G(X_{2^i}) X_{2^i, 2^{i+1}} + D^2 G(X_{2^i}) S_{2^i, 2^{i+1}} + \tfrac{1}{2} D^2 G(X_{2^i}) [\mathbf{X}]_{2^i, 2^{i+1}} \big)$$

where $\int \nabla G(X_r)\, d\mathbf{X}_r$ is well-def by Sewing Lem. and reduced Chen's. With $\int_s^t D^2 G(X_r)\, d[\mathbf{X}]_r$ is defined in Young.