Rough Diff. Equations

(1) <u>Composition with regular func.</u>:

Consider RDE: $dY = f(Y)\,d\mathbf{X}$. $\mathbf{X} \in \mathscr{C}^\gamma$. $I = [0,T]$

Sol. to it will be controlled RP (Y, Y') w.r.t. $\mathbf{X}$. Note that first we need to ensure $f(Y)$ is controlled w.r.t X

<u>Lem¹</u>: $X \in C^\gamma(I, V)$. $(Y, Y') \in \mathscr{D}_X^{2\gamma}(I, W)$. $\varphi \in C_b^2(W, \bar{W})$. Then: $(\varphi(Y), \varphi'(Y)) \in \mathscr{D}_X^{2\gamma}(I, \bar{W})$

where $\varphi'(Y) = D\varphi(Y)Y'$.

<u>Pf</u>: Note $\varphi(Y), \varphi'(Y) \in C^\gamma$

And $R^{\varphi(Y)}_{s,t} = \varphi(Y)_{s,t} - D\varphi(Y_s)Y'_s X_{s,t}$

$= \varphi(Y)_{s,t} - D\varphi(Y_s)Y_{s,t} + D\varphi(Y_s)R^Y_{s,t}$

By Taylor's: $R^{\varphi(Y)} \in C_2^{2\gamma}(I, W)$.

<u>Lem²</u>. Under cond above with $\exists\, m > 1$. st.

$|Y'_0| + \|Y, Y'\|_{X,2\gamma} \le m$. Then: $\exists\, C = C(T, \gamma)$. st.

$\|\varphi(Y), \varphi'(Y)\|_{X,2\gamma} \le C(m+1)\|\varphi\|_{C^2}(1 + \|X\|_\alpha)^2 \cdot$

$(|Y'_0| + \|Y, Y'\|_{X,2\gamma})$

For $\bar{X} \in \mathcal{C}^{\gamma}$. $(\bar{Y}, \bar{Y}') \in \mathcal{D}_{\bar{X}}^{2\gamma}$. st. $M \geq |\bar{Y}_0'|$ $+ \|\bar{Y}, \bar{Y}'\|_{X,2\gamma}$. We also have:

$\|\varphi(Y), \varphi(Y)'; \varphi(\bar{Y}), \varphi(\bar{Y})'\|_{X,\bar{X},2\gamma} \leq C(M,\varphi,T)$.

$(\|X-\bar{X}\|_{\alpha} + |Y_0 - \bar{Y}_0| + |Y_0' - \bar{Y}_0'| + \|Y, Y'; \bar{Y}, \bar{Y}'\|_{X,\bar{X},2\gamma})$

and $C_M \uparrow$ if $M \uparrow$

Pf: Note $\varphi(Y)'_{s,t} = (D\varphi(Y_t) - D\varphi(Y_s))Y_t +$
$D\varphi(Y_s)Y_{s,t}$

with expression of $R^{\varphi(Y)}$ above.

$\Rightarrow \|\varphi(Y), \varphi(Y)'\|_{X,2\gamma} \leq \|D^2\varphi(Y)\|_{\infty}\|Y'\|_{\infty}\|Y\|_{\alpha} +$

$\|D\varphi(Y)\|_{\infty}\|Y\|_{\alpha} + \frac{1}{2}\|D^2\varphi\|_{\infty}\|Y\|_{\alpha}^2 + \|D\varphi\|_{\infty}\|R^Y\|_{2\gamma}$

And we replace $\|Y\|_{\alpha} \leq C(1+\|X\|_{\alpha})(|Y_0'| +$ $T^{\gamma}\|Y, Y'\|_{X,2\gamma})$ by Lem. before.

Similar arguement on latter part.

(2) Solution to RDE:

For $I = [0, T]$. V, W Banach space. $\mathbf{X} \in \mathcal{C}^{\gamma}$ for $\gamma \in (\frac{1}{3}, \frac{1}{2}]$. $\xi \in W$. $f \in C_B^2(W, L(V,W))$.

Def: Sol. to $dY = f(Y)d\mathbf{X}$ with initial datum ξ is a controlled RP $(Y, Y') \in \mathcal{D}_X^{2\gamma}(I, W)$. St.

$Y_t = \xi + \int_0^t f(Y_s)\, d\mathbf{X}_s$, $\forall t \in I$. where $\int f(Y)\, d\mathbf{X}$
$= \int (f(Y), Df(Y)Y')\, d\mathbf{X}$.

<u>Rmk</u>: i) $f(Y)'$ is a canonical choice and is well-def by Lem.

ii) If $V = \mathbb{R}^1$. $W = \mathbb{R}^d$. $X \in C^1$. Then:

$\int f(Y_s)\, d\mathbf{X}_s = \int f(Y_s)\, dX_s$ in sense of RS-integral (even $= \int f(Y_s) X_s'\, ds$) and won't depend on Y' or $f(Y)'$.

So $\forall g \in C^\alpha$, $(Y, g) \in \mathcal{D}_X^{2\alpha}$ since $t \mapsto \int_0^t f(Y_s) X_s'\, ds \in C^1$ as well.

$\Rightarrow$ Its RDE sol. set will be empty or uncountably many.

We need to fix Y' for uniqueness.

iii) For the RDE sol. (Y, Y'). We have:

$(Y, f(Y)) \in \mathcal{D}_X^{2\alpha}(I, W)$ since

$|Y_{s,t} - f(X_s) X_{s,t}| = |\int_s^t f(Y_u)\, d\mathbf{X}_u - \Box|$

$\leq |f(Y_s)' \mathbb{X}_{s,t}| + C|t-s|^{3\alpha}$.

from construction of Rough integral.

We first prove some estimate which can be used to prove continuity of sol. (Lyon's map)

Remark: $\|\cdot\|_{\alpha,J}$ is $\|\cdot\|_\alpha$ only restricted on $J \subset I$.

$$\|\cdot\|_{\alpha,h} := \sup_{J \subset I,\, |J| \le h} \|\cdot\|_{\alpha,J}. \quad J = [s,t] \subset I.$$

prop. (Priori bound for RDE sol.)

For $\xi \in W$, $f \in C_B^2(W, L(V,W))$, $\mathbf{X} \in \mathscr{C}^\alpha(I,W)$ for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. If $(Y, Y') \in \mathcal{D}_X^{2\alpha}(I, W)$ is sol. to the RDE in I with datum ξ with $Y' = f(Y)$. Then: $\exists\, C = C(\alpha) > 0$, s.t.

$$\|Y\|_\alpha \le C\left\{ (\|f\|_{C_B^2} |||\mathbf{X}|||_\alpha) \vee (\|f\|_{C_B^2} |||\mathbf{X}|||_\alpha)^{\frac{1}{\alpha}} \right\}$$

Pf: By scaling on $\mathbf{X}$, WLOG, let $\|f\|_{C_B^2} \le 1$.

We first estimate R^Y:

$$|R^Y_{s,t}| = \left|\int_s^t f(Y_u)\, d\mathbf{X}_u - f(Y_s) X_{s,t}\right|$$

$$\le |\square - Df(Y_s) Y_s' \mathbb{X}_{s,t}| + |Df(Y_s) Y_s' \mathbb{X}_{s,t}|$$

$$\overset{Y' = f(Y)}{\lesssim} (\|X\|_\alpha \|R^{f(Y)}\|_{2\gamma} + \|\mathbb{X}\|_{2\alpha} \|f(Y)'\|_\alpha) |t-s|^{3\gamma}$$

$$+ \|\mathbb{X}\|_{2\alpha} |t-s|^{2\gamma}.$$

from construction of RI.

So: $\|R^Y\|_{2\alpha,h} \lesssim \|\mathbb{X}\|_{2\alpha} + C\|X\|_{\alpha,h}\|R^{f(Y)}\|_{2\alpha,h}$

$\qquad\qquad + \|\mathbb{X}\|_{2\alpha,h}\|f(Y)\|_{\alpha,h})h^{\alpha}$. $(*)$.

Relate R^Y and $R^{f(Y)}$:

$$R^{f(Y)}_{s,t} = f(Y_\cdot)_{s,t} - Df(Y_s)Y_s' X_{s,t}$$

$$= f(Y_\cdot)_{s,t} - Df(Y_s)Y_{s,t} + Df(Y_s)R^Y_{s,t}.$$

$$\|R^{f(Y)}\|_{2\alpha,h} \overset{\text{Taylor}}{\le} \frac{1}{2}|D^2 f|_\infty \|Y\|^2_{\alpha,h} + |Df|_\infty \|R^Y\|_{2\alpha,h}$$

$$\lesssim \|Y\|^2_{\alpha,h} + \|R^Y\|_{2\alpha,h}.$$

With $\|f(Y)\|_{\alpha,h} \lesssim \|Y\|_{\alpha,h}$. Plug into $(*)$:

$$\|R^Y\|_{2\alpha,h} \le C_0(\|\mathbb{X}\|_{2\alpha} + \|X\|_{\alpha,h}\|Y\|^2_{\alpha,h}h^\alpha + \|X\|_{\alpha,h}\|R^Y\|_{2\alpha,h}h^\alpha$$

$$+ \|\mathbb{X}\|_{2\alpha,h}\|Y\|_{\alpha,h}h^\alpha). \quad C_0 = C(T,f).$$

Choose $h = h(\alpha, f, \mathbf{X}) > 0$ suff. small s.t.

$C_1\|X\|_\alpha h^\alpha \le \frac{1}{2}$. $C_1\|\mathbb{X}\|^{\frac{1}{2}}_{2\alpha}h^\alpha \le \frac{1}{2}$. So:

$$\|R^Y\|_{2\alpha,h} \le C_1\|\mathbb{X}\|_{2\alpha} + \frac{1}{2}\|Y\|^2_{\alpha,h} + \frac{1}{2}\|R^Y\|_{2\alpha,h} + \frac{1}{2}\|\mathbb{X}\|^{\frac{1}{2}}_{2\alpha}\cdot\|Y\|_{\alpha,h}$$

With $\|\mathbb{X}\|^{\frac{1}{2}}_\square\|Y\|_\square \overset{\text{Am-Gm}}{\le} \frac{1}{2}\|\mathbb{X}\|_\square + \frac{1}{2}\|Y\|^2_\square$.

$$\Rightarrow \|R^Y\|_{2\alpha,h} \le (2C_1+1)\|\mathbb{X}\|_{2\alpha} + 2\|Y\|^2_{\alpha,h}.$$

Now we plug the estimate of R^Y into:

$\|Y\|_{\alpha,h} \lesssim \|X\|_\alpha + \|R^Y\|_{2\alpha,h}h^\alpha$, So:

multiply $C_2 h^\alpha$ on both sides,

Set $\psi_h := C_2 \|Y\|_{\alpha,h} h^\alpha$. $\lambda_h = C_2 |||X|||_\alpha h^\alpha$.

$\Rightarrow \psi_h \le \lambda_h + \psi_h^2$.

First, we chose h_0 small enough. s.t. $\lambda_h < \frac{1}{4}$. $\forall h < h_0$

(so h will depend on X)

$\Rightarrow \psi_h \ge \psi_+ := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_h} \ge \frac{1}{2}$ or $\psi_h \le \psi_- := \frac{1}{2} - \sqrt{\frac{1}{4} - \lambda_h}$

Note if h small enough (depend on Y) s.t.

$\psi_h \le \frac{1}{2} \Rightarrow \psi_h \le \lambda_h + \psi_h/2$. So: $\psi_h \le 2\lambda_h \xrightarrow{h \to 0} 0$

$\Rightarrow$ We're in second regime.

So we see for h small enough. (say $h < \tilde{h}_0$):

$\psi_h \le \psi_- < 1/6$. for $\forall h < \tilde{h}_0$

With $\|Y\|_{\alpha,h} \le 3 \|Y\|_{\alpha,\frac{h}{3}} \overset{g \ge \frac{h}{3}}{\le} 3 \left(\lim_{g \uparrow h} \|Y\|_{\alpha,g} \wedge \lim_{g \downarrow h} \|Y\|_{\alpha,g}\right)$

$\Rightarrow \psi_h \le 3 \left(\lim_{g \uparrow h} \psi_g \wedge \lim_{g \downarrow h} \psi_g\right)$. $\forall h$. So: $\psi_h < \frac{1}{2}$.

i.e. We will never jump out of regime: $\psi_h \le \psi_-$

So: $\psi_h \le \psi_- = \lambda_h \Big/ \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_h} \le 2\lambda_h$. $\forall h < \tilde{h}_0$.

$\Rightarrow \|Y\|_{\alpha,h} \le C_6 |||X|||_\alpha$. $\forall h < \tilde{h}_0$.

$\|Y\|_{\alpha,h} \le C_2 \|X\|_\alpha + C_2 \|X\|_{2\alpha}^{\frac{1}{2}} + C_2 \|Y\|_{\alpha,h}^2 h^\alpha$.

since $\lambda_h \le \tilde{C} \Rightarrow \tilde{h}_0 \le C_7 |||\mathbf{X}|||_\alpha^{-\frac{1}{\alpha}}$. With Lem.:

<u>Lem.</u> For $\alpha \in (0,1]$, $h > 0$, $M > 0$, $Z : I = [0,T] \to V$

s.t. $\|Z\|_{\alpha,h} \le M \Rightarrow \|Z\|_{\alpha,I} \le M(1 \vee 2h^{-(1-\alpha)})$

<u>Pf:</u> i) If $|t-s| \le h \Rightarrow \frac{\|Z_t - Z_s\|}{|t-s|^\alpha} \le \|Z\|_{\alpha,h} \le M$.

ii) If $|t-s| > h$. Set (t_i) partition of

$[s,t]$. s.t. $|t_i - t_{i-1}| = \delta \le h$.

$\Rightarrow \|Z_t - Z_s\| \le \sum_1^{N-1} \|Z_{t_i} - Z_{t_{i-1}}\| \le \sum M\delta^\alpha$

$\le M N \delta^\alpha$.

Note $N = (t-s)/\delta$. $N \le \frac{t-s}{h} + 1 \le 2\frac{t-s}{h}$

$\Rightarrow \|Z_t - Z_s\| / |t-s|^\alpha \le M \cdot 2^{1-\alpha} \cdot h^{-(1-\alpha)}$.

So: $\|Y\|_\alpha \le C_6 |||\mathbf{X}|||_\alpha (1 \vee 2 C_7 |||\mathbf{X}|||_\alpha^{\frac{1-\alpha}{\alpha}})$

<u>Thm</u> (local well-posedness for RDEs)

$I = [0,T]$. $\xi \in W$. $f \in C^3(W, L(V,W))$. $\mathbf{X} \in \mathscr{C}^\alpha$

(I, V) for some $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Then:

$\exists\, 0 < T_0 \le T$ and unique s.l. $(Y, Y') \in \mathscr{D}_X^{2\alpha}([0,T_0],$

$W)$ with datum ξ on $I_0 = [0, T_0]$ with

$Y' = f(Y)$. Besides if $f \in C_B^3 \Rightarrow T_0 = T$

Rmk: i) Y can either be extended to the whole I or on some max interval $[0,\tau)$ $\subset I$. Besides τ depends on $\mathbf{g}.\mathbf{f}.$ $\mathbf{X}$. In fact. for fix $\mathbf{g}.\mathbf{f}.$ $\mathbf{X} \mapsto \tau(\mathbf{X})$ is l.s.c. i.e. $\underline{\lim}\, \tau(\mathbf{X}_n) \geq \tau(\mathbf{X})$. $\mathbf{X}_n \xrightarrow{\rho_\gamma} \mathbf{X}$.

And if V is finite-dim:

$\lim_{t \uparrow \tau} |Y_t| = +\infty$. i.e. explosion time.

(It doesn't work on infinite-dim)

ii) 3-diff can be seen as "2+1" where "2" is for well-def and "1" is to guarantee uniqueness.

We still have existence for $f \in C^2$.

Rmk: Consider more general RDE:

$dY_t = g(t, Y_t)dt + f(t, Y_t)d\mathbf{X}_t$. where f: $I \times W \to L(V, W)$. $g: I \times W \to W$.

Set $\bar{Y}_t = (Y_t, t) \in W \times I$. $\bar{X}_t = (X_t, t)$.

$$\bar{\mathbb{X}} = \begin{pmatrix} \mathbb{X} & \int X_{s,r}\, dr \\ \int (r-s)\, dX_r & \int (r-s)\, dr \end{pmatrix} \in (V \times I) \otimes (V \times I).$$

And $$\bar{f} = \begin{pmatrix} g(t,w) & 0 \\ 0 & f(t,w) \end{pmatrix} \in \mathcal{L}(I \times W, \overline{I \times W})$$

Then: $d\bar{Y} = \bar{f}(\bar{Y})\,d\bar{\mathbf{X}}$.

So to apply the THM. We require $f, g \in C^3(I \times U)$.

<u>Pf</u>: Assume $f \in C_B^3$ first. Let $T=1$, $\frac{1}{3} < \beta < \alpha = \frac{1}{2}$.

$\forall (Y, Y') \in \mathcal{D}_x^{2\beta}(I, W)$. We have

$(\Xi, \Xi') = (f(Y), Df(Y)Y') \in \mathcal{D}_x^{2\beta}(I, L(U,W))$

For $0 < T \leq 1$. We define:

$\mathcal{M}_T : (Y, Y') \mapsto (\xi + \int_0^{\cdot} \Xi_s \, d\mathbf{X}_s, \Xi)$

So: fixed pt of $\mathcal{M}_T \Longleftrightarrow$ it's sol. to RDE with datum ξ. Since the regularity of (Y, Y') is from $\mathbf{X} \in \mathscr{C}^\alpha$ and

$|Y_{s,t}| = |\int_s^t f(Y_r)\,d\mathbf{X}_r| \overset{f \in C_b}{\lesssim} |X_{s,t}| + \|Y'\|_\infty |\mathbb{X}_{s,t}| + |t-s|^{3\alpha}$

$\Rightarrow Y \in C^\alpha(I, W)$.

Also $R^Y \in C_2^{2\alpha}(I, W)$ by construct of RI.

$\mathcal{M}_T$ can be seen on

$\mathcal{D}_x^{2\beta}(I, W; \xi) := \mathcal{D}_x^{2\beta}(I, W) \cap \{Y_0 = \xi, Y_0' = f(\xi)\}$.

Which's affine space of Banach space $\mathcal{D}_x^{2\beta}$

So it's complete.

Next we want to restrict on unit ball B_T center on $[t \mapsto (\xi + f(\xi) X_{0,t}, f(\xi))] \in \mathcal{D}_X^{2\beta}(I, W; \xi)$

Rmk: Note $[t \mapsto (\xi, f(\xi))] \notin \mathcal{D}_X^{2\beta}(I, W; \xi)$ in fact since $R_{s,t} = f(\xi) X_{s,t} \notin C_2^{2\alpha}$.

$B_T = \{ |Y_0 - \xi| + |Y_0' - f(\xi)| + \| (Y_t - (\xi + f(\xi) X_{0,t}), Y_t' - f(\xi)) \|_{X, 2\alpha} = \| (Y_t - (\xi + f(\xi) X_{0,t}), Y_t' - f(\xi)) \|_{X, 2\alpha} \le 1 \}$.

With $\| (f(\xi) X_{0,t}, f(\xi)) \|_{X, 2\beta} = \| f(\xi) \|_\beta + \| 0 \|_{2\beta} = 0$.

and triangle inequi.:

$B_T = \mathcal{D}_X^{2\beta}([0,T], W; \xi) \cap \{ \| Y, Y' \|_{X, 2\beta} \le 1 \}$.

So $\forall (Y, Y') \in B_T \Rightarrow |Y_0'| + \| Y, Y' \|_{X, 2\beta} \le |f|_\infty + 1 \stackrel{\Delta}{=} M$

Rmk: For $f \in C^3$ only, the bdd M will then depend on ξ.

Next, we want to prove:

a) For $T > 0$ suff. small, $M_T(B_T) \subset B_T$

b) For $T > 0$ suff. small, M_T is contraction on B_T

Lem. $\| f \|_{\infty, [0,T]} \le |f_0| + T^\alpha \| f \|_{\alpha, [0,T]}$.

a) By estimate: $(\| \Xi, \Xi' \|_{X, 2\beta} \leftarrow \| Y \|_\infty)$

$\| \Xi, \Xi' \|_{X, 2\beta} \le C (M+1) \| f \|_{C_b^3} (|Y_0'| + \| Y, Y' \|_{X, 2\beta})$

$\| \int_0^t \Xi_s \, d\mathbf{X}_s, \Xi_t \|_{X,2\beta} \le \| \Xi \|_\beta$

$+ \|\Xi'\|_\infty \|\mathbb{X}\|_{2\beta} + C (\|X\|_\beta \|R^{\Xi}\|_{2\beta} + \|\mathbb{X}\|_{2\beta} \|\Xi'\|_\beta) T^\beta$

(where the 2^{nd}-part is from estimate β-RI.)

Note $2\alpha \le 3\beta$. $T \le 1$. With:

$\|X\|_\beta + \|\mathbb{X}\|_{2\beta} \le T^{\alpha-\beta} \|X\|_\alpha + T^{2\alpha-2\beta} \|\mathbb{X}\|_{2\alpha} \le C T^{\alpha-\beta}$

$\| \int_0^t \Xi_s \, d\mathbf{X}_s, \Xi_t \|_{X,2\beta} \le \|\Xi\|_\beta + C (|\Xi_0'| + \|\Xi, \Xi'\|_{X,2\beta}) T^{\alpha-\beta}$

$\overset{\text{Lem.}}{\le} \|f\|_{C_b^2} \|Y\|_\beta + C \{ \|f\|_{C_b^2}^2 +$

$C(M+1) \|f\|_{C_b^3} (|Y_0'| + \|Y, Y'\|_{X,2\beta}) \} T^{\alpha-\beta}$

Since $\|\Xi\|_\beta \le \|f\|_{C_b^2} \|Y\|_\beta$, $|\Xi_0'| = |Df(Y_0) Y_0'| \le \|f\|_{C_b^2}^2$

With: $|Y_{s,t}| \le |Y'|_\infty |X_{s,t}| + \|R^Y\|_{2\beta} |t-s|^{2\beta}$

$\overset{\text{Lem.}}{\le} C (|Y_0'| + \|Y'\|_\beta) \|X\|_\alpha |t-s|^\alpha + \square$

And $\|R^Y\|_{2\beta} \le \|Y, Y'\|_{X,2\beta} \le 1$. $|Y_0'| + \|Y'\|_\beta \le M$.

$|t-s|^{2\beta-\beta} \le T^\beta \le T^{\alpha-\beta}$ by $2\beta > \alpha$. $T \le 1$.

$\Rightarrow \|Y\|_{\beta,[0,T]} \le C(f) T^{\alpha-\beta}$.

So: $\| \int_0^t \Xi_s \, d\mathbf{X}_s, \Xi_t \|_{X,2\beta,[0,T]} \le C(f, \mathbf{X}, Y, \beta) T^{\alpha-\beta}$.

$\Rightarrow$ We can choose T small enough s.t.

$\|M_T(Y, Y')\|_{X,2\beta} \le 1$. $\forall (Y, Y') \in B_T \Rightarrow \supset M_T(B_T)$

b) We'll show: $(Y, Y'), (\tilde{Y}, \tilde{Y}') \in B_T$, $T \in (0,1)$.

$\|M_T(Y, Y') - M_T(\tilde{Y}, \tilde{Y}')\|_{X,2\beta}$

$$\leq C_f \|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{X,2\beta} T^{\alpha-\beta}.$$

Then shrink T s.t. $C_f T^{\alpha-\beta} < 1 \Rightarrow$ contraction.

Set $h_s := f(Y_s) - f(\tilde{Y}_s)$.

$\|M_T(Y, Y') - M_T(\tilde{Y}, \tilde{Y}')\|_{X,2\beta} = \|\int_0^{\cdot} h_s \, d\mathbf{X}_s, h\|_{X,2\beta}$

$\leq \|h\|_\beta + C(|h_0'| + \|(h, h')\|_{X,2\beta}) T^{\alpha-\beta}$. (estimate a))

$\leq C\|f\|_{C_B^2} \|Y - \tilde{Y}\|_\beta + C\|(h, h')\|_{X,2\beta} T^{\alpha-\beta}$.

Since $h_0' = 0$ (same datum)

Similarly replace Y by $Y - \tilde{Y}$ in estimate a)

$\|Y - \tilde{Y}\|_\beta \leq \|Y' - \tilde{Y}'\|_\beta \|X\|_\alpha T^{\alpha-\beta} + \|R^Y - R^{\tilde{Y}}\|_{2\beta} T^{\alpha-\beta}$

$$\leq C\|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{X,2\beta} T^{\alpha-\beta}.$$

Also, we want to estimate $\|(h, h')\|_{X,2\beta}$:

$h_s = G_s \cdot (Y_s - \tilde{Y}_s)$. $G_s = \int_0^1 Df(tY_s + (1-t)\tilde{Y}_s)\,dt$

$$= g(Y_s, \tilde{Y}_s).$$

We see $g \in C_B^2$ from $f \in C_B^3$. $\|g\|_{C_B^2} \leq \|f\|_{C_B^3}$

So with Lem 1, Lem 2 in (1)

$\Rightarrow \|G, G'\|_{X,2\beta} \leq C_f \|f\|_{C_B^3}$ on B_T.

Lem. $\mathcal{D}_X^{2p}$ is an algebra. i.e. if $G, H \in \mathcal{D}_X^{2p}$.

then: $(GH, (GH)') \in \mathcal{D}_X^{2p}$. $(GH)' = G'H + GH'$.

$$\|GH, (GH)'\|_{X,2p} \le C(|G_0| + |G_0'| + \|G, G'\|_{X,2p}) \cdot$$

$$(|H_0| + |H_0'| + \|H, H'\|_{X,2p})$$

Let $G = G$ where $H = Y - \tilde{Y}$. So $H_0 = H_0' = 0$.

So: $\|(h, h')\|_{X,2p} \le C_f \|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{X,2p} (\|g\|_\infty +$

$$\|g\|_{C_B^1} (|Y_0'| + |\tilde{Y}_0'|) + \|f\|_{C_B^3})$$

$$\le C_f \|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{X,2p}$$

plug both into $\|M_T(Y, Y') - M_T(\tilde{Y}, \tilde{Y}')\|_{X,2p}$

$$\le C_f \|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{X,2p} T^{\alpha - \beta}$$

So for small enough $T \in (0, 1]$. $\exists$ unique sol. (Y, Y') to the SDE by fix pt Thm.

Since T is indep of datum ξ since $f \in C_B^3$

Let $T_i = Ti$. $\xi_i = Y(Ti)$. We can extend it to the whole $[0, T]$.

Rmk: T depend on $\|f\|_{C_B^3}$. If $f \in C^3$ only.

$\lim_{i \to \infty} \tau_i < T$ may happen.

(3) Continuity of Itô-Lyon map:

We want to investigate the Itô-Lyon map

$\hat{S}: \mathscr{C}^{\alpha}(I, V) \to C^{\alpha}(I, W)$ $\quad I = [0, T]$. $Y(\mathbf{X})$

$\quad \mathbf{X} \mapsto Y(\mathbf{X})$ $\quad$ is RDE sol. above.

Rmk: We can also consider $\hat{S}(\mathbf{X}) := (Y, f(Y))(\mathbf{X})$, i.e. controlled RP-valued. But note that $\hat{S}(\mathbf{X})$, $\hat{S}(\bar{\mathbf{X}})$ will stay in different Banach space. $\mathscr{D}_X^{2\alpha}$ & $\mathscr{D}_{\bar{X}}^{2\alpha}$. (We can still intro. $\|\cdot\|_{X, \bar{X}, 2\alpha}$ as dist.)

Thm. (local Lip. of Lyon's map)

If $f \in C_b^3$, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $\mathbf{X}, \bar{\mathbf{X}} \in \mathscr{C}^{\alpha}(I, V)$

$(Y, f(Y))$, $(\bar{Y}, f(\bar{Y}))$ are unique sol. to respective RDE with datum $\xi, \bar{\xi}$.

Assume $M \geq |||\mathbf{X}|||_{\alpha} \vee |||\bar{\mathbf{X}}|||_{\alpha}$. Then:

$\|(Y, f(Y)); (\bar{Y}, f(\bar{Y}))\|_{X, \bar{X}, 2\alpha} \leq$

$C(|\xi - \bar{\xi}| + \rho_{\alpha}(\mathbf{X}, \bar{\mathbf{X}}))$. $C = C(\alpha, f, M) \uparrow$ if $M \uparrow$

(So: $\|Y - \bar{Y}\|_{\alpha} \leq C(|\xi - \bar{\xi}| + \rho_{\alpha}(\mathbf{X}, \bar{\mathbf{X}}))$)

Rmk: For $X = \bar{X}$. We obtain global Lip. of $\hat{S}$ w.r.t datum ξ.

Pf: 1) Set $(\Xi, \Xi') = (f(Y), f(Y)')$

$$(Z, Z') = (\xi + \int f(Y)\, dX, f(Y))$$

$\Rightarrow \|Y, Y'; \bar{Y}, \bar{Y}'\|_{X,\bar{X},2\alpha} = \|Z, Z'; \bar{Z}, \bar{Z}'\|_{X,\bar{X},2\alpha}$

$\le C_0 (\ell_\alpha(X, \bar{X}) + |Df(\xi)f(\xi) - Df(\bar{\xi})f(\bar{\xi})| +$

$T^\alpha \|\Xi, \Xi'; \bar{\Xi}, \bar{\Xi}'\|_{X,\bar{X},2\alpha})$

from stab. on rough integral.

And with $|Df(\xi)f(\xi) - Df(\bar{\xi})f(\bar{\xi})| \le C_f |\xi - \bar{\xi}|$

Combine stab. of $\|\Xi, \Xi'; \bar{\Xi}, \bar{\Xi}'\|_{X,\bar{X},2\alpha}$ again

$\Rightarrow \|Y, Y'; \bar{Y}, \bar{Y}'\|_{X,\bar{X},2\alpha} \le C_1 (1 + T^\alpha C_2) \cdot$

$(\ell_\alpha(X, \bar{X}) + |\xi - \bar{\xi}| + T^\alpha \|Y, f(Y); \bar{Y}, f(\bar{Y})\|_{X,\bar{X},2\alpha})$

$C_1 = C_f \max\{ |||X|||_\alpha, |||\bar{X}|||_\alpha, 1 + \|\Xi, \Xi'\|_{X,2\alpha}, \|\bar{\cdots}\| \}$

$C_2 = C_{f,\alpha} \max\{ 1 + \|Y, Y'\|_{X,2\alpha}, 1 + \|\bar{\cdots}\|_{\bar{X},2\alpha} \}$.

Apply Lem² of (1) on $\|\Xi, \Xi'\|_{X,2\alpha}$. $\|\bar{\Xi} \cdots\|_{\bar{X},2\alpha}$

We see the const. multiple will be C_3

$= C(\alpha, f, T, \|Y, Y'\|_{X,2\alpha}, \|\bar{Y}, \bar{Y}'\|_{X,2\alpha}, |||X|||_\alpha, |||\bar{X}|||_\alpha)$

2) Next, we prove: $\|Y, Y'\|_{X,\alpha} \le C(\alpha, f, T, |||X|||_\alpha)$

$C(T$ if $|||X|||_T)$ and same for $\|\bar{Y},\bar{Y}'\|_{\bar{X},2\gamma}$.

So we can choose T_0 satisfies:

$T_0^{\gamma}C(\gamma, f, T_0, |||X|||_\alpha, |||\bar{X}|||_\alpha) < 1/2 \Rightarrow$ get result.

Then iteratively finite many times to get it on $[0,T]$ since Const. is indept of $\xi, \bar{\xi}$.

And similarly obtain $\|Y-\bar{Y}\|_\alpha$ part.

First by aprior estimate of RDE sol.:

$\|Y'\|_\gamma \le C_f \|Y\|_\alpha \le C(f, T, |||X|||_\alpha) < \infty$.

$|R^Y_{s,t}| \le |f(Y_s)\mathbb{X}_{s,t}| + C(\|X\|_\alpha \|R^{f(Y)}\|_{2\gamma} + \|\mathbb{X}\|_{2\alpha}\|f(Y)'\|_\gamma)|t-s|^{3\gamma}$

$\|f(Y)'\|_\gamma \le C_f \|Y\|_\alpha \le C(f, \gamma, |||X|||_\alpha)$

$\|R^{f(Y)}\|_{2\gamma} \le C_f(\|Y\|_\alpha^2 + \|R^Y\|_{2\gamma})$ from 1^{st} prop of (2)

So: $\|R^Y\|_{2\gamma} \le C(f, T, \gamma, |||X|||_\alpha)(1 + C(f, \gamma, T, |||X|||_\alpha) + \|R^Y\|_{2\gamma} T^{\gamma})$

We can choose $T > 0$ s.t.

$C(f, T, \gamma, |||X|||_\alpha)T^{\gamma} < 1/2 \Rightarrow \|R^Y\|_{2\gamma} \le C(f, T, \gamma, |||X|||_\alpha)$

$\Rightarrow \|Y, Y'\|_{X,2\gamma} = \|Y'\|_\gamma + \|R^Y\|_{2\gamma} \le C(f, T, \gamma, |||X|||_\alpha)$

(4) <u>Connection of RDE and SDE</u>:

Fix B $\mathbb{R}^d$ - BM on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, \mathbb{P})$.

Recall for $f \in C_B(\mathbb{R}^m; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^m)) \cap \text{Lip}$, $\xi \in \mathbb{R}^m$:

$dX_t = f(X_t)\,dB_t$, $X_0 = \xi$ will have unique strong sol. (same for $dX_t = f(X_t) \circ dB_t$)

And for $f \in C_B^3$, a.s. ω - defined RDE:

$dY_t = f(Y_t)\,d\mathbb{B}_t^{z}(\omega)$, $Y_0 = \xi$ will also have unique sol. $Y = Y(\mathbb{B}^z(\omega)) = \hat{S}(\mathbb{B}^z(\omega))$.

<u>Thm.</u> For $f \in C_B^3(\mathbb{R}^m; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^m))$ and $\xi \in \mathbb{R}^m$ determ.

Then: $X_t = \hat{S}(\mathbb{B}^z)$ $\mathbb{P}$-a.s. (Same for Stratonovich case.)

<u>Rmk:</u> i) In this result: we know the sol. to SDE $X_t(\omega) = \hat{S} \circ Y(B(\omega))$, i.e. it will depend on B pathwise ω. (Note it's not clear when consider $X_t = (\int_0^t f(X_s)\,dB_s)(\omega)$ formally)

ii) Since outside null set N, RDE has unique sol. only depend on B rather ξ $\overset{\text{Thm}}{\Rightarrow}$ $\exists$ null set N s.t. indept of ξ, $\exists$ unique strong sol.

$X(\xi)$ to the SDE for $\forall \xi$. (it doesn't work generally since ξ is uncountable)

So: $(\xi, w, t) \mapsto X_t(\xi, w)$ is a.s-well-def flow. i.e. $X_t(\xi, w) = X_t(X_s(\xi), w)$. $\forall t \geq s$.

The price here is regularity: $f \in C_B^3$

iii) We can construct SDE-sol. by fixing $w \in \Omega$ and solving RDE. Then give randomness back to RDE-sol.

Pf: Since there's consistency between rough integral and stochastic integral. We only need to check $Y = \hat{S}(\varphi(B)) \in \mathcal{F}_t$.

From $\varphi: B(w) \mapsto (B(w), \mathbb{B}^2(w))$ is measur. and $\hat{S}$ is conti.

Cor. (Wong-Zakai approxi.)

If B^n is linear pathwise approxi. of B. $f \in C_b^3$. Then sol. X^n to $dX_t^n = f(X_t^n) dB_t^n$ Converge to sol. to $dX_t = f(X_t) \circ dB_t$.

Pf: Recall $(B^n, \int B^n \circ dB^n) \xrightarrow{\mathcal{E}^\alpha} (B, \mathbb{B}^S)$.

With Thm above. X^n a.s. equals to

sol. $Y^n(w)$ to RDE $dY_t^n = f(Y_t^n)\, dB_t^n(w)$

So let $n \to \infty$. $\Rightarrow$ $dY_t = f(Y_t)\, dB^S(w)$

by conti. of Lyon map $\hat{S}$.