Rough Integration

Thm (Sewing Lemma)

$(E, \|\cdot\|)$ is Banach. $A: A_{[0,T]} \to E$. conti.

$\delta A_{s,u,t} \triangleq A_{s,t} - A_{s,u} - A_{u,t}$. $0 \le s \le u \le t$.

If $\exists \lambda > 0$ and $\varepsilon > 0$, s.t. $\|\delta A_{s,u,t}\| \le \lambda |t-s|^{1+\varepsilon}$

Then $\exists Y: [0,T] \to E$, $Y_0 = 0$. Conti. s.t.

$\| Y_s - Y_t - A_{t,s} \| \le C\lambda |t-s|^{1+\varepsilon}$. $C = C(\varepsilon)$

Besides $\lim\limits_{|\pi| \to 0} \sum\limits_{[s,t] \in \pi} A_{s,t} = Y_t - Y_s$.

Pf: By Binomial approxi.: $A^n_{s,t} \triangleq \sum\limits_{0}^{2^n - 1} A_{t_i^n, t_{i+1}^n}$.

where $t_i^n = s + \frac{i}{2^n}(t-s)$.

1) $\| A^n_{s,t} - A^{n+1}_{s,t} \| = \| \sum\limits_{0}^{2^n-1} \delta A_{t_i^n, \frac{t_i^n + t_{i+1}^n}{2}, t_{i+1}^n} \|$

$$\le \lambda \sum_{0}^{2^n-1} |t_{i+1}^n - t_i^n|.$$

$$= \lambda |t-s|^{1+\varepsilon} 2^{-n\varepsilon}$$

So: $\sum\limits_{k}^{n} \| A^n_{st} - A^{n+1}_{st} \| \lesssim 2^{-k\varepsilon} \to 0$.

i.e. (A^n_{st}) is Cauchy $\longrightarrow I_{st}$

Besides $\| I_{st} - A^k_{st} \| = \lim\limits_{n} \| \sum\limits_{k}^{n} A^i_{st} - A^{i+1}_{st} \|$

$$\le \lambda |s-t|^{1+\varepsilon} 2^{-k\varepsilon} / 1 - 2^{-\varepsilon}.$$

2) I_{st} is conti. since it's uniform limit of conti func. A^n_{st}

And $I_{st} = I_{su} + I_{ut}$ by limit argument.

Set $Y_t - Y_s = I_{st}$.

3) $\| Y_t - Y_s - \sum_0^{N-1} A_{t_i, t_{i+1}} \| \overset{1)}{\leq} \frac{\lambda}{1-2^{-\varepsilon}} \sum_0^N |t_{i+1} - t_i|^{1+\varepsilon}$

$\leq \frac{\lambda}{1-2^{-\varepsilon}} |t-s| |\pi|^{\varepsilon} \to 0$.

(1) Young Integral:

1) <u>Prop.</u> $X \in C^{\alpha}$. $Y \in C^{\beta}$. $\alpha \cdot \beta \in (0,1]$. st. $\alpha + \beta > 1$.

Then: $\int_0^t Y_s dX_s := \lim_{\pi \to 0} \sum_{\substack{[s,t] \\ \in \pi}} Y_s X_{st}$ exists

indept of any choice of partition π

Besides $| \int_s^t Y_r dX_r - Y_s X_{st} | \leq C \|X\|_{\alpha} \|Y\|_{\beta} |t-s|^{\alpha+\beta}$

<u>Pf:</u> $A_{st} \triangleq Y_s X_{st}$. $\Rightarrow$ $\delta A_{s,u,t} = - Y_{su} X_{ut}$

So: $\| \delta A_{s,u,t} \| \leq \|Y\|_{\beta} \|X\|_{\alpha} |t-s|^{\alpha+\beta}$.

Apply sewing Lemma. to obtain results.

<u>Rmk:</u> i) We call $\int_0^t Y_s dX_s$ is Young integral of Y_s w.r.t. X_s

ii) Actually. $\lim_{\pi \to 0} \sum_{[u,v]} Y_r X_{uv} = \lim_{\pi \to 0} \sum_{[u,v]} Y_u X_{uv}$.

where v is arbitrary point in $[u,v]$.

Since $|\sum_{[u,v]} Y_{uv} X_{uv}| \leq \|Y\|_\beta \|X\|_\alpha \sum |v-u|^{\alpha+\beta}$

$$\lesssim T |\pi|^{\alpha+\beta-1} \to 0.$$

<u>prop.</u> (Stability)

$X, \tilde{X} \in C^\alpha$. $Y, \tilde{Y} \in C^\beta$. $\alpha, \beta \in (0,1]$. $\alpha+\beta > 1$.

Then $\|\int_0^\cdot Y_s dX_s - \int_0^\cdot \tilde{Y}_s d\tilde{X}_s\|_\alpha \lesssim_{\alpha,\beta,T} (|Y_0 - \tilde{Y}_0| + \|Y-\tilde{Y}\|_\beta)\|X\|_\alpha$

$$+ (|\tilde{Y}_0| + \|\tilde{Y}\|_\beta) \|X - \tilde{X}\|_\beta$$

<u>pf:</u> Next, we will use $|ab - \tilde{a}\tilde{b}| \leq |a||b-\tilde{b}| + |\tilde{b}||a-\tilde{a}|$

repeatedly. in the proof.

Set $A_{st} = Y_s X_{st}$. $\tilde{A}_{st} = \tilde{Y}_s \tilde{X}_{st}$. $\Delta = A - \tilde{A}$.

Apply sewing Lemma. on A_{st}: we have.

$|\int_s^t Y_u dX_u - \int_s^t \tilde{Y}_u d\tilde{X}_u - (Y_s X_{st} - \tilde{Y}_s \tilde{X}_{st})|$

$\lesssim (\|Y-\tilde{Y}\|_\beta \|X\|_\alpha + \|\tilde{Y}\|_\beta \|X - \tilde{X}\|_\alpha) |t-s|^{\alpha+\beta}$

With: $|Y_s X_{st} - \tilde{Y}_s \tilde{X}_{st}| \leq (\|Y-\tilde{Y}\|_\infty \|X\|_\alpha + \|Y\|_\infty \cdots) |t-s|^\alpha$

$\|\tilde{Y}\|_\infty \leq |\tilde{Y}_0| + T^\beta \|Y\|_\beta$. (similar for $Y - \tilde{Y}$).

$\Rightarrow$ We obtain $|\int_s^t Y_r dX_r - \int_s^t \tilde{Y}_r dX_r| \leq$

$|Y_s X_{st} - \tilde{Y}_s \tilde{X}_{st}| + D \leq \cdots$

<u>Lemma</u> (Integrate by part.)

Under the conditions above:

$X_T Y_T = X_0 Y_0 + \int_0^T X_u dY_u + \int_0^T Y_u dX_u$

Pf: $X_T Y_T - X_0 Y_0 = \lim_{|\pi|\to 0} \Sigma (Y_t X_t - Y_s X_s)$

$= \lim_{|\pi|\to 0} \Sigma (Y_s X_{st} + X_s Y_{st} + X_{st} Y_{st})$

With $\| \Sigma X_{st} Y_{st} \| \le \|X\|_\alpha \|Y\|_\beta T |\pi|^{\alpha+\beta-1} \to 0$.

Cor. $X \in C^\alpha([0,T]; V)$. $f \in C^{1+\gamma}(V; \mathbb{R}^1)$. $\alpha, \gamma \in (0,1]$.

s.t. $\alpha + \gamma > 1$. Then: $f(X_T) = f(X_0) + \int_0^T Df(X_s) dX_s$.

Pf: $f(X_T) - f(X_0) = \lim_{|\pi|\to 0} \Sigma\, Df(X_r) X_{st}$. $r \in [s,t]$

Lemma. (Associaty)

$X \in C^\alpha$. $Y, k \in C^\beta$. $\alpha, \beta \in (0,1]$, $\alpha + \beta > 1$. If

$Z \triangleq \int_0 k_s dX_s$. Then $Z \in C^\alpha$. and we have:

$\int_0^T Y_u dZ_u = \int_0^T Y_u k_u dX_u$.

Pf: $\int_s^t Y_u dZ_u = Y_s k_s X_{st} + O(|t-s|^{\alpha+\beta})$

$= \int_s^t Y_s k_s dX_s + O(|t-s|^{\alpha+\beta})$

Taking $\lim_{\pi\to 0} \Sigma_{[s,t]\in\pi}$ on both sides.

Def: $V^p([0,T], \mathcal{U}) := \{ X : [0,T] \to \mathcal{U} \mid (\sup_\pi \Sigma \|X_s - X_t\|^p)^{\frac{1}{p}}$

$=: \|X\|_{p\text{-var}, [0,T]} < \infty \}$. p-variation space.

Rmk: i) $C^\alpha \subsetneq V^{1/\alpha}$. (e.g. step func.)

ii) $V^p = \{ \text{const} \}$. for $\forall p \in (0,1)$.

Actually, we can also define Young integral for p-variation functions:

<u>Prop.</u> (Young-Lōeve estimate)

If $x \in \mathcal{V}^1([0,T], W)$, $\eta \in \mathcal{V}^1([0,T], \mathcal{L}(W,V))$, $1/p + 1/q > 1$

Then: for $I_{st} := \int_s^t \eta_u (dx_u) - \eta_s (x_{s,t})$, we have:

$\exists\, \theta > 1$, s.t. $|I_{st}| \leq \|x\|_{Vp} \|\eta\|_{Vq} / (1-2^{1-\theta})$.

<u>Rmk.</u> i) $\theta = 1/p + 1/q$

ii) For $\mathcal{V}^1$ function, we can define $\int \eta \, dx$ by RS-integral as usual.

<u>Thm.</u> $\forall x \in \mathcal{V}^p([0,T], W)$, $\eta \in \mathcal{V}^q([0,T], \mathcal{L}(W,V))$,

s.t. $\theta = 1/p + 1/q > 1$. Then $\exists (x^n, \eta^n) \subset \mathcal{V}^1 \times \mathcal{V}^1$,

s.t. $x^n \xrightarrow{n} x$, $\eta^n \xrightarrow{n} \eta$. $\sup_n \|x^n\|_{Vp} + \sup_n \|\eta^n\|_{Vq} < \infty$, and $\int_0^{\cdot} \eta_s^n (dx_s^n) \xrightarrow{n} Z =: \int \eta \, dx$.

uniquely. We call Z is the Young integral of η w.r.t. x.

<u>Cor.</u> i) $\left| \int_s^t \eta \, dx - \eta_s x_{st} \right| \leq \|x\|_{Vp} \|\eta\|_{Vq} / (1-2^{1-\theta})$.

ii) $\left\| \int_0^{\cdot} \eta \, dx \right\|_{Vp} \lesssim_{p,q} \|x\|_{Vp} (\|\eta\|_{Vq} + |\eta_0|)$

Cor. The Young integral is unique. indept of choice of every seq. (x^n, η^n) that satisfies the convergence conditions.

Lemma. (Integrate by part.)

Under conditions in Thm above. We have:

$$\int_0^T \eta_s(dx_s) + \int_0^T (d\eta_s)(x_s) + \eta_0(x_0) = \eta_T(x_T).$$

(2) Rough Integration:

① prop. $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha([0,T], V)$. (Y, Y') $\in \mathscr{D}_X^{2\alpha}([0,T], \mathcal{L}(V,W))$. Then:

i) $\int_0^t Y_s \, d\mathbf{X}_s := \lim_{|\pi| \to 0} \sum Y_u X_{uv} + Y'_u \mathbb{X}_{u,v}$ exists

we call it rough integral of Y w.r.t $\mathbf{X}$.

ii) $\left| \int_0^t Y_s \, d\mathbf{X}_s - Y_s X_{st} - Y'_s \mathbb{X}_{s,t} \right| \lesssim_\alpha C(\|R^Y\|_{2\alpha} \|Y\|_\alpha + \|Y'\|_\alpha \|\mathbb{X}\|_{2\alpha}) |t-s|^{3\alpha}$.

Pf: $A_{st} \triangleq Y_s X_{st} + Y'_s \mathbb{X}_{st}$. Apply sewing Lemma.

Note $|\delta A_{sut}| = \| R^Y_{su} X_{ut} + Y'_{su} \mathbb{X}_{ut} \| \lesssim_\alpha C(\square) |t-s|^{3\alpha}$

Cor. $\mathbf{X} \in \mathcal{C}^\alpha$. $(Y, Y') \in \mathscr{D}_X^{2\alpha}$. $\Rightarrow (\int Y_s \, d\mathbf{X}_s, Y)$ $\in \mathscr{D}_X^{2\alpha}$. is also controlled path.

Rmk: It's like differentiate the rough integral w.r.t X.

Cr. $F \in C^{2\alpha}$. replace $\mathbf{X}$ by $\widetilde{\mathbf{X}} = (X_t, \mathbb{X}_{s,t} + F_{st})$ above. Then: $(Y, Y') \in \mathcal{D}^{2\alpha}_{\widetilde{X}}$, and

$$\int_0^T Y_u \, d\widetilde{\mathbf{X}}_u = \int_0^T Y_u \, d\mathbf{X}_u + \int_0^T Y_u' \, dF_u .$$

Thm (Stability).

$\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{X}, \widetilde{\mathbf{X}} \in \mathcal{C}^\alpha$. $Y, \widetilde{Y} \in \mathcal{D}^{2\alpha}_X, \mathcal{D}^{2\alpha}_{\widetilde{X}}$.

$$\Rightarrow \|Y - \widetilde{Y}\|_\alpha \underset{\alpha, T}{\lesssim} (|Y_0' - \widetilde{Y}_0'| + \|\widetilde{Y}' - Y'\|_\alpha)\|X\|_\alpha + \|X - \widetilde{X}\|_\alpha (|\widetilde{Y}_0'| + \|\widetilde{Y}'\|_\alpha) + \|R^Y - R^{\widetilde{Y}}\|_{2\alpha} T^\alpha .$$

and
$$\|R^{\int Y dX} - R^{\int \widetilde{Y} d\widetilde{X}}\|_{2\alpha} \underset{\alpha, T}{\lesssim} (|Y_0' - \widetilde{Y}_0'| + \|Y' - \widetilde{Y}'\|_\alpha + \|R^Y - R^{\widetilde{Y}}\|_{2\alpha}) \, |||X|||_\alpha + (|Y_0'| + \|\widetilde{Y}'\|_\alpha + \|R^{\widetilde{Y}}\|_{2\alpha}) \, \|\mathbf{X}, \widetilde{\mathbf{X}}\|_\alpha$$

Pf: i) By $|ab - \tilde{a}\tilde{b}| \le |a||b - \tilde{b}| + |\tilde{b}||a - \tilde{a}|$

ii) Apply sewing Lemma on $\Delta_{st} = A_{st} - \widetilde{A}_{st}$. with estimation of $\|Y_s' \mathbb{X}_{st} - \widetilde{Y}_s' \widetilde{\mathbb{X}}_{st}\|$.

Cr. $\|\int_0^\cdot Y_s \, d\mathbf{X}_s - \int_0^\cdot \widetilde{Y}_s \, d\widetilde{\mathbf{X}}_s\|_\alpha \underset{\alpha, T}{\lesssim} (1 + \|X\|_\alpha + \|\widetilde{X}\|_\alpha)$

$(\|\widetilde{Y}, \widetilde{Y}'\|_{\mathcal{D}_X^{2\alpha}} \|X, \widetilde{X}\|_\alpha + (|Y_0 - \widetilde{Y}_0| + \|Y - \widetilde{Y}'\|_\alpha + |Y_0' - \widetilde{Y}_0'|$

$+ \|R^Y - R^{\widetilde{Y}}\|_{2\alpha}) \, |||X|||_\alpha)$

Rmk: We can control distance of $(Y, \tilde{Y})$.

$(\int Y dX, \int \tilde{Y} d\tilde{X})$. and $(R^{\int Y dX}, R^{\int \tilde{Y} d\tilde{X}})$.

Thm. (Associaty).

$\mathbf{X} \in \mathcal{C}^{\gamma}$. (Y, Y'), $(Z, Z') \in \mathcal{D}_X^{2\gamma}$. Then: we have.

$\int_0^t Y_s dZ_s := \lim_{|\pi| \to 0} \sum Y_u Z_{uv} + Y_u' Z_u' \mathbb{X}_{u,v}$ exists

$$\left| \int_s^t Y_u dZ_u - Y_s Z_{st} - Y_s' Z_s' \mathbb{X}_{st} \right| \lesssim_\alpha C(\|Y'\|_\infty \|Z'\|_\alpha \|X\|_\alpha^2$$

$$+ \|Y\|_\alpha \|R^Z\|_{2\alpha} + \|R^Y\|_{2\alpha} \|Z'\|_\infty \|X\|_\alpha + \|Y'Z'\|_\alpha \|\mathbb{X}\|_{2\alpha}) |t-s|^{3\gamma}$$

Rmk: i) Both controlled path looks locally like X

$\Rightarrow$ They look like each other locally.

ii) Note $\forall Z \in \mathcal{D}_X^{2\alpha}$. we can lift it by itself by setting:

$$\mathbb{Z}_{s,t} := \int_s^t Z_{s,r} dZ_r \Rightarrow \mathcal{D}_X^{2\gamma} \xrightarrow{\mathcal{E}} \mathcal{C}^{\gamma}.$$

(By limit argument, it also satisfies Chen's)

Conversely. $\mathbf{X} \in \mathcal{C}^{\gamma}$. $\Rightarrow (X, id) \in \mathcal{D}_X^{2\gamma}$.

(or: $\mathcal{C}^{2\alpha} \hookrightarrow \mathcal{D}_X^{2\alpha}$. by set $\underline{Y} = (Y, 0)$)

Pf: $A_{st} \triangleq Y_{st} Z_{st} + Y_s' Z_s' \mathbb{X}_{st}$

apply sewing Lemma. on A_{st}.

Cor. $(Y, Y'), (K, K') \in \mathscr{D}_X^{2\alpha}$. Set $(Z, Z') = (\int K \, d\mathbf{X}, K) \in \mathscr{D}_X^{2\alpha}$. Then:

$$\int_0^t Y_s \, dZ_s = \int_0^t Y_s K_s \, d\mathbf{X}_s .$$

Pf: As the case of Young integral.

Thm (Consistency).

$\mathbf{X} \in \mathscr{C}^\alpha$, $(Z, Z') \in \mathscr{D}_X^{2\alpha}$. $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}^\alpha$ is canonical lift of Z defined in Rmk above. If $(Y, Y') \in \mathscr{D}_Z^{2\alpha}$. Then: $(Y, Y'Z') \in \mathscr{D}_X^{2\alpha}$ and $\int_0^t Y_s \, d\mathbf{Z}_s = \int_0^t Y_s \, d\mathbf{Z}_s$.

Pf: 1) $Y_{st} - Y_s' Z_s' X_{st} = R_{st}^Y + Y_s' R_{st}^Z \in \mathcal{C}^{2\alpha}$.

2) $LHS \overset{[s,t]}{=} Y_s Z_{st} + Y_s' \mathbb{Z}_{st} + O(|t-s|^{3\alpha})$

$= Y_s Z_{st} + Y_s' Z_s' Z_s' \mathbb{X}_{st} + O(|t-s|^{3\alpha})$

$= \int_s^t Y_r \, dZ_r + O(|t-s|^{3\alpha})$.

④ Itô Formula:

Def: $\mathbf{X} \in \mathscr{C}^\alpha$. The bracket of $\mathbf{X}$ is defined by

$$[\mathbf{X}]_t := X_{0,t} \otimes X_{0,t} - 2\,\mathrm{Sym}(\mathbb{X}_{0,t})$$

Rmk: i) $[\mathbf{X}]_{s,t} = X_{s,t} \otimes X_{s,t} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t})$ is

easy to check by Chen's.

$\Rightarrow [X] \in C^{2\alpha}$ in particular.

ii) $\mathbf{X} \in \mathscr{C}_g^{\alpha} \Leftrightarrow [\mathbf{X}]_t = 0$

iii) Recall $\mathrm{Sym}(\mathbb{B}^{\text{Ito}}_{s,t}) = \frac{1}{2}(B_{st} \otimes B_{st} - (t-s)I)$

$\Rightarrow [(B, \mathbb{B}^{\text{Itô}})]_t = tI$.

Consistent with the common one.

Lemma. $\mathbf{X} \in \mathscr{C}^{\alpha}$. $(K, K') \in \mathscr{D}_X^{2\alpha}$. Set $(Z, Z') =$

$(\int_0^{\cdot} K_u \, d\mathbf{X}_u, K_s) \in \mathscr{D}_X^{2\alpha}$. $\mathbf{Z} = (Z, \mathbb{Z})$ is

canonical lift of Z. Then:

$[\mathbf{Z}]_t = \int_0^t (K_u \otimes K_u) \, d[\mathbf{X}]_u \quad \forall t \in [0,T]$

Rmk: It's alike Itô isometry.

Pf: Note RHS exists as Young integral.

So: $\int_0^T (K_u \otimes K_u) \, d[\mathbf{X}]_u = \lim_{|\pi| \to 0} \sum (K_s \otimes K_s)[X]_{st}$

And $[\mathbf{Z}]_{st} = (K_s \otimes K_s)[\mathbf{X}]_{st} + O(|t-s|^{3\alpha})$.

by replace $Z_{st} = K_s X_{st} + K_s' \mathbb{X}_{st} + O(|t-s|^{3\alpha})$

Prop. (Itô Formula)

$\mathbf{X} \in \mathscr{C}^{\alpha}$. $f \in C^3$. $\Rightarrow f(X_T) = f(X_0) + \int_0^T Df(X_u) \, d\mathbf{X}_u + \frac{1}{2} \int_0^T D^2 f(X_u) \, d[\mathbf{X}]_u$

Pf: WLOG. $f \in C_b^3$. $\Rightarrow (Df(X), D^2f(X)) \in D_X^{2\gamma}$

Note $\sum_{ij} A_{ij} B_{ij} = -\sum A_{ji} B_{ji}$ if A is sym. B is anti-sym. $\Rightarrow \sum A_{ij} B_{ij} = 0$.

So: $D^2 f(X_s) \mathbb{X}_{st} = D^2 f(X_s)\, \mathrm{sym}(\mathbb{X}_{st})$.

$$\Rightarrow f(X_t) - f(X_s) = (Df(X_s) X_{st} + \frac{1}{2} D^2 f(X_s) [\mathbb{X}]_{st}) + R_{st}$$

$$|R_{st}| = \left| \int_0^1 \int_0^1 (D^2 f(X_s + r_1 r_2 X_{st}) - D^2 f(X_s)) (X_{st} \otimes X_{st}) r_1 \, dr_2 \, dr_1 \right|$$

$$\lesssim \|f\|_{C^3} \|X\|_\alpha^3 |t-s|^{3\gamma}. \Rightarrow \lim_{|\pi| \to 0} \sum |R_{st}| = 0$$

Rmk: If we introduce more general $(\beta, \gamma+\beta)$-controlled path $(Y, Y') \in D_X^{\gamma+\beta}$. $\mathbb{X} \in \mathcal{C}^\alpha$.

i.e. $Y \in C^\alpha$. $Y' \in C^\beta$. $R^Y \in C_2^{\gamma+\beta}$. $2\alpha + \beta > 1$.

$\Rightarrow$ It's enough to take $f \in C^{1/\alpha + \varepsilon}$. $\forall \varepsilon > 0$.

Cor. $\mathbb{X} \in \mathcal{C}^\alpha$. $(Y, Y'), (Y', Y'') \in D_X^{2\alpha}$. If

$Y_t = Y_0 + \int_0^t Y_s' \, d\mathbb{X}_s + I_t$. $I \in C^{2\alpha}$.

Then: when $f \in C^3$. we have:

$$f(Y_T) - f(Y_0) = \int_0^T Df(Y_s) Y_s' \, d\mathbb{X}_s + \int_0^T Df(Y_u) \, dI_u$$
$$+ \frac{1}{2} \int_0^T D^2 f(Y_s)(Y_s' \otimes Y_s') \, d[X]_s$$

Pf: By Chain rule. / Cor. of ①.

Def: $X : [0,T] \to \mathbb{R}^d$ has finite QV in sense of Föllmer along partition π if $\forall t, i, j$

$$[X^i, X^j]_t^\pi := \lim_{\pi \to 0} \sum X^i_{u,v} X^j_{u,v} \text{ exists.}$$

Lemma. If $X \in C([0,T], \mathbb{R}^d)$ has finite QV along π in Föllmer sense. Then: $[X]^\pi \in BV[0,T]$, and $\forall G \in C([0,T], L(\mathbb{R}^{d \otimes 2}; \mathbb{R}^n))$. We have:

$$\lim_{\pi \to 0} \sum G(u) X_{u,v} \otimes X_{u,v} = \int_0^t G(u)\, d[X]^\pi_u$$

Thm (Itô - Föllmer)

For $F \in C^2$. If $X \in C([0,T]; \mathbb{R}^d)$, has finite and conti. QV $[X]^\pi$ along π in Föller sense.

Then: $F(X_t) - F(X_0) = \int_0^t DF(X_u)\, dX_u + \frac{1}{2} \int_0^t D^2F(X_u)\, d[X]^\pi_u$

where $\int DF(X)\, dX$ is def in left-point RS sum.

Rmk: It holds for general conti. path.