Consistency with Itô-calculus

(1) Integration:

① Itô Brownian:

prop. On $(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,T]}, \mathbb{P})$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

Consider $\mathbf{B} = (B_t, \mathbb{B}_t^{Itô}) \in \mathscr{C}^{\alpha}$, $\mathcal{F}_t$-adapted

Itô rough path. If $(Y(\omega), Y'(\omega)) \in \mathscr{D}^{2\alpha}_{B(\omega)}$

are adapted. Then: $\int_0^T Y_u \, d\mathbf{B}_u \overset{a.s.}{=} \int_0^T Y_u \, dB_u$.

Pf: Only need to prove: $\sum Y_s' \mathbb{B}_{s,t}^{Itô} \xrightarrow[|\pi|\to 0]{} 0$ a.s.

Note $\int_s^t \mathbb{B}_{s,r} \, dB_r$ is a martingale.

$\Rightarrow \mathbb{E} \left| \sum_{\pi} Y_s' \mathbb{B}_{s,t} \right|^2 \underset{ortho.}{=} \sum \mathbb{E}(|Y_s' \mathbb{B}_{s,t}|^2)$

$\overset{BDG.}{\leq} CM^2 \sum |s-t|^2 \leq CM^2 T |\pi| \to 0$

if we assume $\|Y'\|_\infty \leq M$.

Otherwise, set $\tau_m = T \wedge \inf\{t>0 \mid |Y_t'| \geq m\}$.

replace Y_t by $Y_t^{\tau_m}$. Set $m \to \infty$.

② Stratonovich Brownian:

prop. Replace $\mathbf{B}$ by $(B, \mathbb{B}^{Strat.})$ above. We

also have: $\int_0^T Y_s \, d\mathbf{B}_s = \int_0^T Y_s \circ dB_s$.

Pf: Note $\lim_{|\pi|\to 0} \sum Y_s' (\mathbb{B}_{st}^{\text{Itô}} - \mathbb{B}_{st}^{\text{Strat}})$

$$= -\frac{1}{2} \lim_{|\pi|\to 0} \sum Y_s' (t-s) = -\frac{1}{2} \int_0^T Y_u' \, du.$$

$\Rightarrow$ We prove: $\int_0^T Y_u' \, du = \langle Y, B \rangle_T$ a.s.

$$Y_{st} B_{st} = Y_s' (B_{st} \otimes B_{st}) + R_{st}^Y B_{st}.$$

$$= 2 Y_s' \,\mathrm{Sym}(\mathbb{B}_{st}^{\text{Itô}}) + R_{st}^Y B_{st} + Y_s' (t-s).$$

$\sum Y_s' \,\mathrm{Sym}(\mathbb{B}_{st}^{\text{Itô}}) \to 0$ a.s. by $\sum Y_s' \mathbb{B}_{st}^{\text{Itô}} \xrightarrow{a.s.} 0$.

$|\sum R_{st}^Y B_{st}| \leq \|R^Y\|_{2\alpha} \|B\|_\alpha T |\pi|^{3\alpha - 1} \to 0$

So: $\langle Y, B \rangle_T = \lim_{|\pi|\to 0} \sum Y_{st} B_{st} = \int_0^T Y_s' \, ds$

Prop. For $f \in C_B^3$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$, $y \in L^2(\Omega, \mathcal{F}, P)$.

consider $\mathbf{B} = (B, \mathbb{B}^{\text{Itô}}) / (B, \mathbb{B}^{\text{Strat}}) \in \mathcal{C}^\alpha$.

The solution of $dY_t = f(Y_t) \, d\mathbf{B}_t^{\text{Itô}} / f(Y_t) \, d\mathbf{B}_t^{\text{Strat}}$.

exists uniquely, equals the solution of

$dY_t = f(Y_t) \, dB_t \;/\; f(Y_t) \circ dB_t$.

Pf: Note $\sigma(B_s, \mathbb{B}_{s,t}^{\text{Itô}}, 0 \leq s < t \leq T) = \sigma(B_s, s \leq T)$.

and continuity of Lyons's map. $\Rightarrow (Y_t, Y_t') \in \mathcal{F}_t$.

which satisfies measurability condition.

Thm (Wong - Zakai).

$\alpha < \frac{1}{2}$. For Y^n satisfies $dY^n = f(Y^n) \, dt + \tilde{f}(Y^n) \, dB^n$.

B^n is dyadic approxi. and $dY = f(Y) \, dt + \tilde{f}(Y) \circ dB$.

$\Rightarrow \|Y - Y^n\|_{\alpha, T} \to 0$ a.s. where $\tilde{f} \in C^1$. $f \in C_B^3$