Stochastic Sewing Lemma.

(1) Proof:

Fix $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$. $Z_k \in \mathcal{F}_i$. $\mathbb{R}^d$-valued. $\forall k < i$.

$S^n \triangleq \sum_1^n Z_i$, $S_1^n = \sum_1^n \mathbb{E}(Z_i | \mathcal{F}_i)$, $S_2^n = S^n - S_1^n$

Lemma. $\|S^n\|_{L^m} \leq \sum_1^n \|\mathbb{E}(Z_i|\mathcal{F}_i)\|_{L^m} + 2C_m (\sum_1^n \|Z_i\|_{L^m}^2)^{\frac{1}{2}}$

for $\forall m \geq 2$.

Rmk: Requirement of $m \geq 2$ is from BDG.

Pf: $\|S_1^n\|_m \leq \sum_1^n \|\mathbb{E}(Z_i|\mathcal{F}_i)\|_{L^m}$

$\|S_2^n\|_m \underset{\text{BDG.}}{\leq} C_m \mathbb{E}^{\frac{1}{m}}((\sum_1^n |Z_i - \mathbb{E}(Z_i|\mathcal{F}_i)|^2)^{\frac{m}{2}})$

$\underset{\text{Minkov.}}{\leq} C_m (\sum_1^n \|Z_i - \mathbb{E}(Z_i|\mathcal{F}_i)\|_{L^m}^2)^{\frac{1}{2}}$

$\underset{\text{Minkov.}}{\overset{\text{contract}}{\leq}} 2C_m (\sum_1^n \|Z_i\|_m^2)^{\frac{1}{2}}$.

Thm. For $m \geq 2$. $A: A_{[0,T]} \to L^m$. conti. $A_{s,s} = 0$.

$A_{s,t} \in \mathcal{F}_t$. $\forall s \leq t$. If $\exists \lambda_i, \varepsilon_i > 0$. s.t.

$\|\mathbb{E}(\delta A_{sut})\|_{L^m} \leq \lambda_1 |t-s|^{1+\varepsilon_1}$. $\|\delta A_{sut}\|_{L^m} \leq \lambda_2 |t-s|^{1+\varepsilon_2}$.

Then $\exists$ unique Y_s. s.t. $Y_0 = 0$. $Y_t \in \mathcal{F}_t$. $Y \in L^m$.

i) $\exists C_1, C_2 > 0$. s.t. $|Y_t - Y_s - A_{st}| \leq C_1 |t-s|^{1+\varepsilon_1} + C_2 |t-s|^{\frac{1}{2}+\varepsilon_2}$

$\|\mathbb{E}(Y_t - Y_s - A_{st} | \mathcal{F}_t)\|_{L^m} \leq C_1 |t-s|^{1+\varepsilon_1}$

ii) $\sum_{\pi} A_{u,v} \xrightarrow[|\pi| \to 0]{L^m} Y_t - Y_s$. π is partition of $[s,t]$.

Pf: 1) Dyadic argument as common one:

$$A_{st}^n = \sum_0^{2^n-1} A_{t_i^n, t_{i+1}^n}. \quad \text{Set } u_i^n = (t_i^n + t_{i+1}^n)/2.$$

$$\Rightarrow A_{st}^n - A_{st}^{n+1} = \sum_0^{2^n-1} \delta A_{t_i^n, u_i^n, t_{i+1}^n} := I_1^n + I_2^n.$$

$$= \sum \mathbb{E}(\delta A_\square \mid \mathcal{F}_{t_i^n}) + \sum (\delta A_\square - \mathbb{E}(\cdots))$$

By argument in Lemma above with the conditions in Thm:

$$\| A_{st}^n - A_{st}^{n+1} \|_{L^m} \le \lambda_1 |t-s|^{1+\varepsilon_1} 2^{-n\varepsilon_1} + 2 C_m \lambda_2 |t-s|^{\frac{1}{2}+\varepsilon_2} 2^{-n\varepsilon_2}$$

$\Rightarrow A_{st}^n$ Cauchy in L^m.

Set $I_{st} = \lim_n A_{st}^n$. exists

Check it satisfies all conditions. $\Rightarrow$ let $Y_t - Y_s = I_{st}$.

2) Uniqueness:

If $\bar{Y}, Y$ are two paths satisfying conditions.

Set $\xi = Y - \bar{Y}$. By 1): we have.

$\|\xi_t - \xi_s\|_{L^m} \le \tilde{C} |t-s|^{\frac{1}{2}+\tilde{\varepsilon}}$. $\|\mathbb{E}(\xi_t - \xi_s \mid \mathcal{F}_s)\|_{L^m} \le \tilde{C} |t-s|^{1+\tilde{\varepsilon}}$.

Note $\xi_t = \sum_0^{2^n-1} (\xi_{t_{i+1}^n} - \xi_{t_i^n})$. $\forall n$.

By Lemma. $\Rightarrow \|\xi_t\|_{L^m} \lesssim 2^{-n\tilde{\varepsilon}} \to 0$.

3') Note $Y_t - Y_1 - \sum_0^{N-1} A_{t_i, t_{i+1}} = \sum_0^{N-1} (Y_{t_{i+1}} - Y_{t_i} - A_{t_i, t_{i+1}})$

Apply the Lemma again to obtain ii).

(3) Application on Itô calculus:

i) For B_t is $\mathbb{R}^d$-SBM. $f \in C^\beta(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^n))$. $\beta \le 1$

$\int_0^t f(B_t)\,dB_t := \lim_{\pi \to 0} \sum f(B_s) B_{s,u}$ exists. and

$\| \int_s^t f(B_r)\,dB_r - f(B_s) B_{st} \|_{L^m} \lesssim |t-s|^{\frac{1}{2}+\frac{\beta}{2}}$.

Pf: Apply Stochastic sewing Lemma. on $A_{st} = f(B_s) B_{st}$.

By scaling prop. of BM:

$\| \delta A_{sut} \|_{L^m} \le \| f(B)_{s,u} \|_{L^m} \| B_{ut} \|_{L^m}$

$\le \| f \|_{C^\beta} \| B_1 \|^\beta_{L^{m\beta}} |u-s|^{\frac{\beta}{2}} \| B_1 \|_{L^m} |t-u|^{\frac{1}{2}}$

$\lesssim |t-s|^{\frac{\beta}{2}+\frac{1}{2}}$.

ii) $M \in L^4$. $\mathcal{F}_t$-adapted mart in $\mathbb{R}^d$. If:

$\| M_{su} \otimes M_{ut} \|_{L^2} \le c |t-s|^{\frac{1}{2}+\varepsilon}$. Then $\langle M \rangle_t :=$

$\lim_{\pi \to 0} \sum M_{uv} \otimes M_{uv}$ exists.

Pf: Set $A_{st} = M_{st} \otimes M_{st}$. Then: we have.

$\delta A_{sut} = M_{su} \otimes M_{ut} + M_{ut} \otimes M_{su}$.

So: $\| \delta A_{sut} \|_{L^2} \le 2c |t-s|^{\frac{1}{2}+\varepsilon}$. by SSL.

Rmk: We call $\langle M \rangle$ is QV of M. We also

have: $\| \langle M \rangle_{st} - M_{st} \otimes M_{st} \|_{L^2} \lesssim |t-s|^{\frac{1}{2}+\varepsilon}$.

$\mathbb{E}(M_{st} \otimes M_{st} \mid \mathcal{F}_s) = \mathbb{E}(\langle M_{st} \rangle \mid \mathcal{F}_s)$.

(3) On rough integral:

Prop. For $f \in C_B^2$. $\mathbb{X} \in \mathscr{C}^\gamma$. $\gamma \in (\frac{1}{3}, \frac{1}{2}]$. $B_\cdot$ is BM.

Then: $\int_0^t f(B_r + X_r) \, d\mathbb{X}_r := \lim_{\pi \to 0} \sum f(B_u + X_u) X_{uv} + f'(B_u + X_u) \mathbb{X}_{u,v}$

exists in L^m. $\forall t$.

Besides. $\int_0^\cdot f(B_r + X_r) \, d\mathbb{X}_r$ is unique L^m. $\mathcal{F}_t$-adapted process. s.t.

$\| \int_s^t f(B_r + X_r) \, d\mathbb{X}_r - f(B_s + X_s) X_{st} - Df(X_s + B_s) \mathbb{X}_{st} \|_{L^m} \lesssim |t-s|^{2\gamma}$

$\| \mathbb{E}(\int_s^t f(\cdots) - \cdots - Df(X_s + B_s) \mathbb{X}_{st} \mid \mathcal{F}_s) \|_{L^m} \lesssim |t-s|^{3\gamma}$

Rmk: If f is linear. Then it's common in stochastic calculus. We extend f to general C_B^2 case.

Pf: Set $A_{st} = f(B_s + X_s) X_{st} + Df(B_s + X_s) \mathbb{X}_{st}$

$\delta A_{s,u,t} = -(f(B+X)_{s,u} - Df(B_s + X_s) X_{s,u}) X_{ut} - Df(B+X)_{s,u} \mathbb{X}_{u,t}$.

Note $\| f(B+X)_{s,u} \|_{L^m} \lesssim \|f\|_{C^1} (\|B_1\|_{L^m} |u-s|^{\frac{1}{2}} + \|X\|_\gamma |u-s|^\gamma)$

For the later part. $(\mathbb{E}(\cdots \mid \mathcal{F}_s) \leq \cdots)$

Note $f(B+X)_{s,u} - Df(X_s + B_s) X_{s,u} = Df(B_s + X_s) B_{s,u} + \int_0^1 (Df(B_s + X_s + r(B_{s,u} + X_{s,u})) - Df(B_s + X_s))(B_{s,u} + X_{s,u}) \, dr$.