Motivation

Consider ODE: $dy(t) = f(t, y(t))dt$. $y(0) = y_0$.
we have Picard-Lindelöf Thm to get its solution theory:
f is global Lip. / local Lip. & sublinear growth ($\langle f(x) - f(y)\rangle (x-y) \le C|x-y|^2$). $\Rightarrow$ $\exists$ unique sol.
and $(f, y_0) \in (\mathcal{F} \times \mathbb{R}^d, |\cdot|_\infty + |\cdot|) \mapsto y(t) \in C(I)$.
(locally uniform) is conti. $\mathcal{F}$ is set of f above.
Rmk: Unique fails example: $f(x) = 2\,\mathrm{sgn}(x)\sqrt{|x|}$.

$$y_0 = 0. \Rightarrow y(t) = \pm t^2 \text{ or } y(t) = 0.$$

Next, we replace dt by $dX(t)$ to study more general: $dY_t = f(Y_t)dX_t$. $Y_0 = y_0$.
where X_t is conti: $I \to \mathbb{R}^m$. $f: \mathbb{R}^d \to \mathbb{R}^{d\times m}$.
We want to know its well-posedness and Stability (i.e. $S: (X, y_0) \mapsto Y$ is conti.?)

1) Classical: $X_t \in C^1(I; \mathbb{R}^m)$. $\Rightarrow$ it reduces to

$$dy_t = \tilde{f}(t, y_t)dt. \quad \tilde{f}(t, y) = f(t, y)X'_t.$$

Rmk: It doesn't permit $X \in C^\alpha$. $\alpha < 1$.

2) Young ⊂ Riemann–Stieltjes integral: It permits $Y_t, X_t \in C^\alpha, C^\beta$. s.t. $\alpha+\beta>1$.

(while $\alpha+\beta\le 1$ has counterexample.)

Rmk: Note $Y_{s,t} \approx f(Y_s) X_{s,t}$ in the ODE. $f(Y_s)$ is at most regularity of Y.

So Y inherits the regularity of X. $\Rightarrow$ It requires $X \in C^\alpha$. $\alpha > \frac{1}{2}$ to use Young approach. Then it still excludes BM case

But it will still cause some stab. problem:

Thm. Solution map $S: (C^\infty(I;\mathbb{R}^2) \times \mathbb{R}^2, |\cdot|_\alpha + |\cdot|_{\mathbb{R}^2})$ $\to (X, y_0) \mapsto Y \in (C(I, \mathbb{R}^2), 2pt)$ is disconti.

Pf: Set $f(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ x_1 & 0 \end{pmatrix}$ $Y(0) = (X^1(0), 0)$.

$$X^{(n)}(t) = \left(\frac{\cos(n^2 t)}{n}, \frac{\sin(n^2 t)}{n}\right) \xrightarrow{n\to\infty} 0.$$

So: $Y^{(n),1}(t) = X^{(n),1}(t) = \cos(n^2 t)/n \xrightarrow{n\to\infty} 0$

$$Y^{(n),2}(t) = \int_0^t X^{(n),1}(t)\, dX^{(n),2}(t) \to t/2 \neq 0.$$

$\Rightarrow S(X^{(n)}, X^{(n)}(0)) \not\to 0$. as $(X^{(n)}, X^{(n)}(0)) \to 0$

Rmk: Note that it also implies X_t converges $\not\Rightarrow$

$\int_0^{\cdot} X_r \otimes dX_r = (\int_0^{\cdot} X^i_r \otimes dX^j_r)_{i,j}$ converges

So the correct idea to resolve this is to consider $S: (X, \int X \otimes dX) \in C^1(I; \mathbb{R}^m) \oplus C^1(I \times I; \mathbb{R}^{m \times m}) \mapsto y \in C(I, \mathbb{R}^n)$

And $\overline{\{(X, \int X \otimes dX) : X \in C^1\}}^{d} = \{(X, \mathbb{X}) \in C \times C\}$ space of rough path. (d is suitable metric)

<u>Rmk</u>: $S = \hat{S} \circ \Psi$. $\Psi: X \mapsto (X, \mathbb{X})$. rough path lift. $\hat{S}: (X, \mathbb{X}) \mapsto Y(X, \mathbb{X})$ is solution map of RDE.

<u>Def</u>: (RS Integration)

$X: [0,T] \to V$. $y: [0,T] \to L(V, W)$. $D_n = (t^i_n) \subset [0,T]$. s.t. $|D_n| \to 0$. $\xi^n_i \in [t^i_n, t^{i+1}_n]$. some pt. RS integral $\int_0^T y_u dX_u$ is $\lim_{n \to \infty} \sum y(\xi^n_i) X_{t^i_n, t^{i+1}_n}$ which's indept. of (ξ^n_i) and D_n.

<u>prop</u>. RS integral exists for $X \in C^{1\text{-var}}$ and y piecewise conti. And also:

$|\int_0^T y dX| \leq \|y\|_{\infty, [0,T]} \|X\|_{1\text{-var}, [0,T]}$

prop. Under conditions above, we have:

i) (Integration by part)

$$y_T X_T - y_0 X_0 = \int_0^T y_u \, dX_u + \int_0^T (dy_u) \, dX_u .$$

ii) (Continuity)

$(X, y) \in (C^{1\text{-var}}, \|\cdot\|_{1\text{-var}}) \times (C, \|\cdot\|_\infty) \mapsto$

$\int y \, dX \in (C^{1\text{-var}}, \|\cdot\|_{1\text{-var}})$ is BLO.

And if $\sup_n \|X^n\|_{1\text{-var}, [0,T]} < \infty$, $X^n \xrightarrow{\|\cdot\|_\infty} X$ with

$y^n \xrightarrow{\|\cdot\|_\infty} y$. $\Rightarrow \int y^n dX^n \xrightarrow{\|\cdot\|_\infty} \int y \, dX$.

Def: (Young integration)

For $X \in C^{p\text{-var}}([0,T], V)$, $y \in C^{q\text{-var}}([0,T], L(V, W))$

$z \in C([0,T], W)$ is Young integral of y over

X, if $\exists (X^n), (y^n) \subset C^{1\text{-var}}$, s.t.

a) $|X^n - X|_\infty \to 0$, $\sup |X^n|_{p\text{-var}} < \infty$.

b) $|y^n - y|_\infty \to 0$, $\sup |y^n|_{q\text{-var}} < \infty$.

c) $\int_0^{\cdot} y^n dX^n \to z$ uniformly on $[0,T]$.

Rmk: z may not be unique.

prop. When $p^{-1} + q^{-1} > 1$. Then z is unique

so well-def. And it will be

consistent with def : $\lim_{n\to\infty} \sum \eta(s_i^n) \cdot X_{t_i^n, t_{i+1}^n}$. i.e. limit of Rieman sum.

We also have Young-Loeve estimate:

$|Z_{s,t} - \eta_s X_{s,t}| \le C(p,q) |X|_{p\text{-var},[s,t]} |\eta|_{q\text{-var},[s,t]}$.

<u>prop.</u> (Continuity)

Under conditions above, we have

i) $(X,\eta) \in C^{p\text{-var}} \times C^{q\text{-var}} \mapsto \int_0^\cdot \eta\, dX \in C^{p\text{-var}}$

is BLO.

ii) $p^{-1} + q^{-1} > 1$. $(X^n), X \in C^{p\text{-var}}$. $(\eta^n), \eta \in C^{q\text{-var}}$

If $\sup |X^n|_{p\text{-var}}, \sup |\eta^n|_{q\text{-var}} < \infty$. with

$|X^n - X|_\infty \to 0$, $|\eta^n - \eta|_\infty \to 0$. Then.

$\lim_{n\to\infty} |\int_0^\cdot \eta^n dX^n - \int_0^\cdot \eta\, dX|_{\infty,[0,T]} = 0$. &

$\sup_n |\int_0^\cdot \eta^n dX^n|_{p\text{-var},[0,T]} < \infty$.

3') Stochastic integration: $\int f(s)\, dX_s$ defined for Semi-mart. X_s and random field $f(s)$ in the sense of Itô or Stratonovich. But there're some flaws:

i) It's only defined $\mathbb{P}$-a.s. rather giving the meaning to $w \mapsto \int f(s,w)\, dX_s(w)$ for $\forall w$.

ii) For fixed (y_0, f), $w \in \Omega = C(I; \mathbb{R}^d) \mapsto Y(w)$ isn't conti. (even not well-def.)

iii) X need to be semi-mart. Then fBMs or other rough signal process don't work.

4°) Lebesgue-Stieltjes integral:

It's more generally defined for right-conti. BVs / Radon measure. But it has totally different way to define and it lacks of regularity.