Spaces of Rough Paths

<u>Denote</u>: i) $I = [0, T]$. V is separable Banach space with norm $|\cdot|_V$. ($|\cdot|$ is $\mathbb{R}^d$-norm)

ii) Seminorm $\|X\|_\alpha := \sup_{s,t \in [0,T]} |X_{s,t}| / |t-s|^\alpha$ and norm $\|X\|_{C^\alpha} := |X_0| + \|X\|_\alpha$.

iii) $C_2^\alpha(I^2, W)$ is space of two-parameter processes $\mathbb{X}$ s.t. $\|\mathbb{X}\|_\alpha < \infty$.

(1) <u>Hölder conti. RPs</u>:

<u>Def</u>: $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. A V-valued α-Hölder conti. rough path $\mathbf{X} = (X, \mathbb{X})$ satisfies:

i) (Regularity) $X \in C^\alpha(I, V)$, $\mathbb{X} \in C_2^{2\alpha}(I, V^{\otimes 2})$

ii) (Chen's relation) $\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}$ for $\forall s, u, t \in I$.

Denote space of them by $\mathscr{C}^\alpha(I, V)$

<u>Rmk</u>: i) Two param process $\mathbb{X} : I^2 \to V^{\otimes 2}$ is additive if $\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t}$. Note that $\delta X_{s,t} := X_t - X_s$ is additive. But rough

component $\mathbb{X}$ doesn't! So we see that Chen's relation isn't linear.

ii) Take $s=u=t$. $\Rightarrow \mathbb{X}_{t,t}=0$. $\forall t \in I$.

iii) Take $u=t$. $\Rightarrow \mathbb{X}_{s,t} = -\mathbb{X}_{t,s} - X_{s,t} \otimes X_{t,s}$

$= X_{s,0} \otimes X_{0,t} + \mathbb{X}_{s,0} + \mathbb{X}_{0,t}$

$= -\mathbb{X}_{0,s} + \mathbb{X}_{0,t} - X_{0,s} \otimes X_{0,t} + X_{0,s} \otimes X_{0,s}$

So $t \mapsto (X_{0,t}, \mathbb{X}_{0,t})$ already determines $\mathbb{X}$.

(The two para. $\mathbb{X}$ can be considered as one para. path)

iv) For $V=\mathbb{R}^1$. $X \in C^\alpha \Rightarrow \mathbb{X}_{s,t} := \frac{1}{2}(X_{s,t})^2$ $\in C_2^{2\alpha}$ also satisfies Chen's relation

So: $X \in C^\alpha(I;\mathbb{R}^1)$ can be lifted to rough path $\mathbf{X}$ in $\mathscr{C}^\alpha$.

For general V. it holds by Lyons-Victoir extension Thm for $\forall \alpha \leq 1$.

<u>Lem</u>: $X \in C(I,V)$. $\mathbb{X}_{s,t} := \int_s^t X_{s,r} \otimes dX_r$ is defined by any kind integration (e.g. RS..)

s.t. $\int_s^t = \int_s^u + \int_u^t$. $\int_s^t dX_r = (X_{s,t}$. and

$f \mapsto \int_s^t f\, dX_r$ is Linear. Then: We have $(X, \mathbb{X})$ satisfies Chen's relation.

Rmk: One should think $\mathbb{X}_{s,t}$ is substitute of $\int_s^t X_{s,r} \otimes dX_r$ when $\alpha \le \frac{1}{2}$.

Lem^2: For $s = z_0 < z_1 \cdots < z_N = t$. Then. Chen's relation implies: $\mathbb{X}_{s,t} = \sum_0^{N-1} (\mathbb{X}_{z_i, z_{i+1}} + X_{s,z_i} \otimes X_{z_i, z_{i+1}})$.

Pf: By induction on N.

Lem^3 (Uniqueness)

$X \in C^\alpha$. $(X, \mathbb{X})$. $(X. \bar{\mathbb{X}}) \in \mathscr{C}^\alpha$. Then: $\mathbb{X}_{s,t} - \bar{\mathbb{X}}_{s,t} = G_{s,t} \in C_2^{2\alpha}(I^{\otimes 2}; V^{\otimes 2})$ s.t. G is additive. Conversely for $\forall G \in C^{2\alpha}(I, V^{\otimes 2})$ We have $(X, \mathbb{X} + G) \in \mathscr{C}^\alpha$.

Pf: $G \in C_2^{2\alpha}$ is clear. And for additive:

$$G_{s,t} = \mathbb{X}_{s,t} - \bar{\mathbb{X}}_{s,t} \overset{\text{Chen's}}{=} (\bar{\mathbb{X}}_{0,s} - \mathbb{X}_{0,s}) - (\bar{\mathbb{X}}_{0,t} - \mathbb{X}_{0,t})$$

Converse is easy to check.

Cor. For $(X, \mathbb{X}) \in \mathscr{C}^\alpha$. $\{\bar{\mathbb{X}} : (X. \bar{\mathbb{X}}) \in \mathscr{C}^\alpha\}$

$$= \{\mathbb{X} + \delta G \mid G \in C^{2\alpha}(I, V^{\otimes 2})\}.$$

Def: $\mathscr{L}(C^\alpha) := \{(X, \mathbb{X}) \in \mathscr{C}^\alpha : X \in C^\infty. \mathbb{X}_{s,t} =$

$\int_s^t X_{s,r} \otimes dX_r \}$ space of canonical RPs

$\mathscr{C}^\infty := \{ (X, \mathbb{X}) \in \mathscr{C}^\alpha : X \in C^\infty, \mathbb{X} \in C_2^\infty \}$.

space of smooth RPs.

<u>Rmk</u>: By Lem above: $\mathscr{L}(C^\infty) \subsetneq \mathscr{C}^\infty \subsetneq \mathscr{C}^\alpha$.

<u>e.g.</u> For $X \equiv 0$. $\Rightarrow (0, 0) \in \mathscr{L}(C^\infty)$. But:

$(0, \delta G) \in \mathscr{C}^\infty \notin \mathscr{L}(C^\infty)$. $\forall G \in C^\infty$.

For $X \in C^\alpha(I, \mathbb{R}^1)$. $(X, \frac{1}{2}(\delta X)^2) \in \mathscr{C}^\alpha$.

but not $\in \mathscr{C}^\infty$.

<u>Rmk</u>: Why $\alpha \in (\frac{1}{3}, \frac{1}{2}]$?

If $\alpha > \frac{1}{2}$, $X \in C^\alpha$. $\mathbb{X}$ is its iterated Young integral. $\forall \bar{\mathbb{X}}$. s.t. $(X, \bar{\mathbb{X}}) \in \mathscr{C}^\alpha$.

$\Rightarrow \bar{\mathbb{X}} = \delta G + \mathbb{X}$. $G \in C^{2\alpha}(I, V^{\otimes 2})$. $2\alpha > 1$.

So: $\delta G = 0$. i.e. $\mathbb{X}$ is unique.

Then: $\mathbb{X}$ can only be $\int X \otimes dX$ Young integral. Reduce the case to Young sense.

(2) <u>Norms and Metrics</u>:

Equip $C^\alpha(I, V) \oplus C_2^{2\alpha}(I, V^{\otimes 2})$ with $\|\cdot\|_{C^\alpha} + \|\cdot\|_{C^{2\alpha}}$.

Note $\mathscr{C}^\alpha \subset C^\alpha \oplus C_2^{2\alpha}$. But due to Chen's relation, $\mathscr{C}^\alpha$ isn't LS / n.v.s. i.e. For $(X, \mathbb{X})$, $(Y, \mathbb{Y}) \in \mathscr{C}^\alpha \not\Rightarrow (X+Y, \mathbb{X}+\mathbb{Y}) \in \mathscr{C}^\alpha$.

(Since it doesn't satisfy Chen's relation)

Rmk: But for $X+Y=Z$, $\Rightarrow Z \in \mathscr{C}^\alpha$. There exists lift of $Z \neq \mathbb{X}+\mathbb{Y}$ by Lyon's Thm.

Def: $\delta_\lambda : (X, \mathbb{X}) \mapsto (\lambda X, \lambda^2 \mathbb{X})$ homo scaling

Set δ_λ-homo. norm on $\underline{X} \in C^\alpha \oplus C_2^{2\alpha}$ is:

$|||\underline{X}|||_{C^\alpha} = \|X\|_{C^\alpha} + \sqrt{\|\mathbb{X}\|_{2\alpha}}$.

Rmk: $\mathscr{C}^\alpha$ is metric space with metric induced by $|||\cdot|||_\alpha$. (but not LS).

$\rho_{C^\alpha}(\underline{X}, \underline{Y}) := \|X - Y\|_{C^\alpha} + \|\mathbb{X} - \mathbb{Y}\|_{2\alpha}$ is called α-Hölder rough path metric

Prop. i) $(\mathscr{C}^\alpha, \rho_{C^\alpha})$ is complete metric space.

ii) For $V = \mathbb{R}^1$, $(\mathscr{C}^\alpha(I; \mathbb{R}^1), \rho_{C^\alpha})$ is not separable.

iii) (Interpolation) For $\frac{1}{3} < \alpha < \beta \le \frac{1}{2}$, $(\underline{X}^n) \subset \mathscr{C}^\beta$. If $\sup_n |||\underline{X}^n|||_\beta \le C_0 < \infty$ with

$X^n \to X$, $\mathbb{X}^n \to \mathbb{X}$ pointwise. Then: $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\beta$. But $\rho_\alpha(\mathbf{X}^n, \mathbf{X}) \to 0$ only for $\alpha < \beta$ rather β.

Rmk: "$X^n \to X$" can be replaced by "$\delta X^n \to \delta X$" in iii), which's weakened one.

e.g. $X^n = 1 \not\to X = 0$. but $\delta X^n = 0 \to \delta X = 0$.

Pf (iii) 1) First assume $\delta X^n, \mathbb{X}^n$ both converge uniformly on $\forall (s,t) \in I^2$.

$\Rightarrow X \in C^\beta$, $\mathbb{X} \in C^{2\beta}$

Besides, $\mathbf{X}$ satisfies Chen's relation follows from pointwise convergence.

By assumpt of uniform convergence:

$\exists \varepsilon_n$. $|X^n_{s,t} - X_{s,t}| \le \varepsilon_n$. also $\le 2C_0 |t-s|^\beta$

Similarly. $|\mathbb{X}^n_{s,t} - \mathbb{X}_{s,t}| \le \varepsilon_n$. $2C_0 |t-s|^{2\beta}$

By interpolation: $(a \wedge b \le a^\theta b^{1-\theta})$

$$\Rightarrow \begin{array}{l} |X^n_{s,t} - X_{s,t}| \le C \varepsilon_n^{1-\frac{\alpha}{\beta}} |t-s|^\alpha \\ |\mathbb{X}^n_{s,t} - \mathbb{X}_{s,t}| \le C \varepsilon_n^{1-\frac{\alpha}{\beta}} |t-s|^{2\alpha} \end{array}$$

So: $\rho_\alpha(\mathbf{X}^n, \mathbf{X}) \le C \varepsilon_n^{1-\frac{\alpha}{\beta}} \to 0$. $(n \to \infty)$

2) To prove X^n & $\mathbb{X}^n$ will converge uniformly: Let $D=\{t_i\}$ partition s.t. $C_0 \max_i |t_i - t_{i+1}|^{\beta} \le \varepsilon/8$ and $|D| < \infty$.

So: $|X_{s,t} - X^n_{s,t}| \le |X_{\bar{s},\bar{t}} - X^n_{\bar{s},\bar{t}}| + |X_{s,\bar{s}}|$
$+ |X_{t,\bar{t}}| + |X^n_{\bar{s},s}| + |X^n_{\bar{t},t}| \le$ (β-Hölder)
$|X^n_{\bar{s},\bar{t}} - X_{\bar{s},\bar{t}}| + \frac{\varepsilon}{2} \le \varepsilon$

And similar argument for $\mathbb{X}^n \in C_2^{2\tau}$.

(3) <u>Geometric RPs</u>:

Note for $\mathbf{X} \in \mathscr{L}(C^{\tau}(\mathbb{R}^d))$, it satisfies:

$$\mathbb{X}^{ij}_{s,t} + \mathbb{X}^{ji}_{s,t} = X^i_{s,t} X^j_{s,t} \Rightarrow \mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$$

where $\mathrm{Sym}(M) = \frac{1}{2}(M + M^T)$. $M \in \mathrm{Mat}_{d\times d}$.

For general V: set $(e_i) \subset V^*$. basis.

$$e_i^* \otimes e_j^*(\mathbb{X}_{s,t}) + e_j^* \otimes e_i^*(\mathbb{X}_{s,t}) = e_i^*(X_{s,t})\, e_j^*(X_{s,t})$$

is called geometric relation.

<u>Def</u>: $\mathscr{C}_g^{\tau}(I, V)$ is weakly geometric τ-Hölder RP space. i.e. collect all the $\mathscr{C}^{\tau}$-RPs satisfying geometric relation

And $\mathscr{C}_g^{0,\alpha} := \overline{\mathcal{L}(C^\infty)}^{\,\varrho_\alpha}$ denote α-Hölder geometric RP space.

Rmk: i) $\mathscr{C}_g^{0,\alpha} \subsetneq \mathscr{C}_g^{\alpha} \subsetneq \mathscr{C}^{\alpha}$. (e.g. $\mathbb{B}^{\text{Itô}}$, $\mathbb{B}^{\text{strat}}$)

ii) V is separable $\Rightarrow \mathscr{C}_g^{0,\alpha}$ is separable w.r.t ϱ_α. While $\mathscr{C}_g^{\alpha}$ isn't. So: $\mathscr{C}^{\alpha}(I, V) \supset \mathscr{C}_g^{\alpha}$ isn't.

iii) Distinction of $\mathscr{C}_g^{\alpha}$ and $\mathscr{C}_g^{0,\alpha}$ doesn't matter. For $\beta > \alpha$:

$\mathscr{C}_g^{\beta} \subset \mathscr{C}_g^{0,\alpha}$ (So it still exists smooth approxi. w.r.t ϱ_α for $\mathscr{C}_g^{\beta}$)

(4) Brownian Motion:

Denote B_t is d-dim BM on $I = [0, T]$.

Recall $B_t \in C^{\alpha}$, $\forall \alpha < \frac{1}{2}$.

Apply Lem' of (1) on Itô. Strat integral. We have its iterated Itô/Strat integral satisfy Chen's relation. Beyond that:

Lem. $\mathbb{B}^S$ satisfies geometric relation while $\mathbb{B}^I$ doesn't ($2\,\mathrm{Sym}(\mathbb{B}^I_{s,t}) = B_{s,t} \otimes B_{s,t} - (t-s)I$)

Pf: By Itô's and Levy characterization

Cor: $Ant(\mathbb{B}^I_{s,t}) = Ant(\mathbb{B}^S_{s,t})$, where

$$Ant(M) = \frac{1}{2}(M - M^T) \text{ for } M \in Mat_{d\times d}.$$

Next, we want to check regularity of $\mathbb{B}^S$ and $\mathbb{B}^I$. (i.e. $\mathbb{B}^S, \mathbb{B}^I \in C_2^{2\alpha}$, $\forall \alpha < \frac{1}{2}$).

Prop. (Rough Kolmogorov criterion)

For $q \geq 2$, $\beta > \frac{1}{q}$. If $\forall s,t \in I$, we have:

$|X_{s,t}|_{L^q} \leq C|t-s|^{\beta}$, $|\mathbb{X}_{s,t}|_{L^{q/2}} \leq C|t-s|^{2\beta}$

Then: $\forall \alpha \in [0, \beta - 1/q)$, $\exists$ modification of $(X, \mathbb{X})$ and $K_\alpha \in L^q$, $\mathbb{K}_\alpha \in L^{q/2}$, s.t.

$|X_{s,t}(\omega)| \leq K_\alpha(\omega)|t-s|^{\alpha}$, $|\mathbb{X}_{s,t}(\omega)| \leq \mathbb{K}_\alpha(\omega)|t-s|^{2\alpha}$

for $\forall s,t \in I$, $\forall \omega \in \Omega$.

Pf: 1°) Set $I = [0,1]$, $D_n = \{k/2^n\}_{k=0}^{2^n-1}$ and let

$K_n \triangleq \sup_{D_n} |X_{t,t+2^{-n}}|$, $\mathbb{K}_n \triangleq \sup_{D_n} |\mathbb{X}_{t,t+2^{-n}}|$

$\Rightarrow \mathbb{E}(K_n^q) \leq \sum_{D_n} \mathbb{E}(|X_{t,t+2^{-n}}|^q) \lesssim 2^{n(1-\beta q)}$

Similarly, $\mathbb{E}(\mathbb{K}_n^{q/2}) \lesssim 2^{n(1-\beta q)}$.

For $s<t \in D := \cup_n D_n$, let m s.t. $2^{-m-1} < t-s$ $\leq 2^{-m}$. Write $t = s + \sum_{m}^{N} 2^{-k}\delta_i$, $\delta_i \in \{0,1\}$.

choose $\{z_i\}$. s.t. $s = z_0 < z_1 \cdots < z_m = t$. &

$[z_i, z_{i+1}] \in D_n$. $\forall n$. and D_n will at most

contain two such i.

So: $|X_{s,t}| \leq \max_{0 \leq i \leq m-1} |X_{s,z_{i+1}}| \leq \sum_0^{m-1} |X_{z_i,z_{i+1}}|$

$$\leq 2 \sum_{n \geq m+1} k_n$$

Similarly: $|\mathbb{X}_{s,t}| \overset{\text{Lem 2}}{=} |\sum_0^{m-1} \mathbb{X}_{z_i,z_{i+1}} + X_{s,z_i} \otimes X_{z_i,z_{i+1}}|$

$$\leq 2 \sum_{n \geq m+1} \mathbb{K}_n + (2 \sum_{n \geq m+1} k_n)^2.$$

$|X_{s,t}| / |t-s|^\alpha \overset{|t-s| \leq \square}{\leq} 2 \sum_{m+1} k_n 2^{\alpha(m+1)} \leq 2 \sum_{m+1} k_n 2^{\alpha n}$

$$\leq 2 \sum_{n \geq 1} k_n 2^{\alpha n} =: k_\alpha.$$

$|k_\alpha|_{L^2} \leq 2 \sum 2^{\alpha n} |k_n|_{L^2} \lesssim \sum 2^{\alpha n + n(\frac{1}{2} - \beta)}$

$< \infty$ for $\alpha < \beta - 1/q$.

Similarly. Set $\mathbb{K}_\alpha := 2 \sum \mathbb{K}_n 2^{\alpha n} \in L^{2/2}(\Omega)$.

$|\mathbb{X}_{s,t}| / |t-s|^{2\alpha} \leq \mathbb{K}_\alpha + k_\alpha^2 \in L^{2/2}$ follows by

estimate of $|\mathbb{K}_n|_{L^{2/2}}$. $|k_n|_{L^2}$ above.

2) So $\exists N \in \mathcal{F}$. null set. s.t. X. $\mathbb{X}$ are $\alpha, 2\alpha$

-Hölder conti. $\forall \alpha < \beta - 1/q$ on $\cup D_n$. $\forall w \in N^c$

We can define the unique conti. extension

$\bar{X}, \bar{\mathbb{X}}$ of $X, \mathbb{X}$ on UD_n for $w \in N^c$ and

$\bar{X} = 0$, $\bar{\mathbb{X}} = 0$ for $w \in N$.

$\Rightarrow \bar{X}, \bar{\mathbb{X}} \in C^\alpha, C_2^{2\alpha}$ pathwise. (since they are conti. So $\|\cdot\|_{C^\alpha}$ is determined on UD_n)

3') Check $(\bar{X}, \bar{\mathbb{X}})$ is modification of $(X, \mathbb{X})$.

For $t \notin UD_n$. Note $\bar{X}_t = \lim_{t_n \in UD_n \to t} X_{t_n}$

And we see: $\mathbb{P}(|X_{t_n} - X_t| \geq \eta) \leq \|X_{t_n} - X_t\|_{L^2} / \eta^2$

$\overset{\text{cond.}}{\leq} C|t_n - t|^\beta / \eta^2 \to 0.$

So $X_{t_n} \xrightarrow{pr} X_t \Rightarrow \bar{X}_t = X_t$ a.s. $\forall t$.

<u>Cor.</u> $(B, \mathbb{B}^I)$, $(B, \mathbb{B}^S)$ satisfy the cond. of prop. above with $\beta = \frac{1}{2}$, $q \geq 2$.

<u>Pf:</u> By scaling prop. of BM.

<u>Cor.</u> $B^I, B^S \in \mathscr{C}^\alpha$ a.s. for $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

<u>Cor.</u> $\mathbb{P}(|||B^I|||_{C^\alpha} \geq t) \sim \mathbb{P}(|||B^S|||_{C^\alpha} \geq t) \sim e^{-ct}$ $(t \to \infty)$ for some $c > 0$.

Furthermore, we see $B^S \in \mathscr{C}_g^{0,\beta}$ a.s. $\forall \beta < \frac{1}{2}$.

So $\exists B^n := (B^n, \mathbb{B}^n)$, s.t. $B^n \in \mathcal{L}(C^\infty) \xrightarrow{\mathscr{C}_\beta} B^S$

prop. B^n is n^{th} step piecewise linear approxi. of B i.e. $B^n_t = B_t$ for $t = Ti/2^n$ by linear interpolation. Then:

$\mathbb{B}^n_t := \int_s^t B^n_{s,r} \otimes dB^n_r$ in RS integral sense.

We have: $\rho_\alpha(\mathbf{B}^n, \mathbf{B}^S) \xrightarrow{n\to\infty} 0$. $\forall \alpha < \frac{1}{2}$.

Rmk: But not every reasonable pathwise approxi. of B can be lifted as ρ_α - approxi. of $\mathbf{B}^S$. e.g.

There exists some smooth and reasonable approxi. of Bm. st.

its $\mathcal{L}(C^\infty)$-lift converges to $\bar{\mathbf{B}} = (B, \bar{\mathbb{B}})$. $\bar{\mathbb{B}} = \mathbb{B}^S_{s,t} + (t-s)A$. where $A^T = -A$.

So $\bar{\mathbb{B}} \neq \mathbb{B}^I$ as well.

Since $\mathrm{Ant}(\bar{\mathbb{B}}) \neq \mathrm{Ant}(\mathbb{B}^I) = \mathrm{Ant}(\mathbb{B}^S)$