Introduction

Examples:

① Stochastic filtering:

Signal $dX_t = \sigma(X_t)dB_t + b(X_t)dt + \gamma(X_t)dY_t$. $X_0 \sim \mu$

where $\gamma(x)$ is correlation coefficient.

Observation $dY_t = h(X_t)dt + dB_t^\perp$. where $(B, B^\perp)$ is 2-dim BM under $\mathbb{P}^0$.

The nonlinear filtering problem is to estimate filter $\mathbb{E}^{\mathbb{P}^0}(\varphi(X_t) \mid \mathcal{G}_t^Y)(\omega) = \pi_t(\varphi, \omega)$.

$= \mu_t(\varphi)/\mu_t(1)$. ($\mu_t(1)$ is normalizer)

$\mu_t(\varphi) := \mathbb{E}^{\mathbb{P}}(\varphi(X_t)G_t \mid \mathcal{G}_t^Y)$ where $G_t := d\mathbb{P}^0/d\mathbb{P}|_{\mathcal{G}_t} = \exp(\int_0^t h(X_s)dY_s - \frac{1}{2}\int_0^t h^2(X_s)ds)$

i.e. We change the measure to $\mathbb{P}$. And under $\mathbb{P}$. (B, Y) is 2-dim BM by Girsanov

Besides, μ_t satisfies Zakai SPDE:

$\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_s, A\varphi \rangle ds + \langle \mu_s, \Gamma\varphi \rangle dY_s$

Where A is 2^{nd}-order term of the generator of signal. $\Gamma\varphi := h\cdot\varphi + \gamma\cdot D\varphi$.

i) Existence of sol. for Zakai follows from Feynman-Kac Thm. (First construct $\gamma\cdot D\varphi + \mathcal{L}\varphi$. then add $h\cdot\varphi$ by Feynman-Kac's)

2) Uniqueness of sol. for Zakai requires duality argument.

e.g. (Toy Version)

Assume Y is fixed and smooth.

$\mu_t' = M_t^* \mu_t$, $M_t = A_t + \Gamma_t \cdot D$. $\mu_0 = \mu$.

Consider dual backward PDE:

$-u' = M_t u$. $u_T = \varphi$.

Then: $\langle \mu_T, \varphi \rangle = \langle \mu_T, u_T \rangle \underset{\text{rule}}{\overset{\text{product}}{=}} \langle \mu_0, u_0 \rangle$

$= \langle \mu, u_0 \rangle$. $\forall \varphi \in C_c^\infty$.

So if $\mu, \tilde{\mu}$ satisfies Zakai with same feature. then $\mu_t = \tilde{\mu}_t$. $\forall t$

Rmk: Such argument can be used for rough FPE by replace Y with $\mathbf{Y}$.

② <u>Local Stochastic Volatility</u>:

consider to model log price process X_t:

$dX_t = l(X_t)V_t(w)(\rho_t dW_t + \bar{\rho}_t dB_t)$ where (W, B) is 2-dim BM. V_t is $\mathcal{F}_t^W$-adapted volatility process. ρ, $\bar{\rho}$ are deterministic correlated func. s.t. $\rho^2 + \bar{\rho}^2 = 1$.

If $l \equiv 1$. $\mathcal{L}(X_T \mid \mathcal{F}_T^W) = N(\int_0^T V_t \rho_t dW_t, \int_0^T V_t^2 \bar{\rho}_t^2 dt)$

We want to pricing:

$u(t, y)(w) := \mathbb{E}(g(X_T^{t,y}) \mid \mathcal{F}_T^W)(w)$

It'll satisfy SPDE:

$-\partial_t u = L_t(w) u\, dt + \gamma(t, x) \partial_x u \circ d\overleftarrow{m}_t$. where $L_t(w) + \gamma(t, x) \cdot D$ is generator of $X \mid \mathcal{F}^W$.

<u>Rmk</u>: $\overleftarrow{m}_t$ is backward "mart" here but it may not semimart.

③ McKean-Vlasov SDE:

Consider MFG with common noise:

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)dB_t^i + f(X_t^{N,i}, \mu_t^N)dW_t$$

$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{N,i}} | \mathcal{F}_T^W$. $X_0^{N,i} \sim \mu_0$. $(B^i)_i^N$ is BM.

By propagation of chaos:

$\mathcal{L}((X^{N,1} \dots X^{N,k}) | \mathcal{F}_T^W) \to \mathcal{L}(X | \mathcal{F}_T^W)^{\otimes k}$. where

$$dX_t = b(X_t, \mu_t)dt + \sigma(X_t, \mu_t)dB_t + f(X_t, \mu_t)dW_t$$

$\mu_t = \mathcal{L}(X_t | \mathcal{F}_T^W)$ will also satisfy SPDE

$$d\mu_t = (\tfrac{1}{2}\partial_{ij}(\tilde{a}^{ij}(\cdot, \mu_t)\mu_t) - \partial_i(b^i(\cdot, \mu_t)\mu_t))dt$$
$$- \partial_i(f^i(\cdot, \mu_t)\mu_t)dW_t^i$$

Rmk: It can also be extended to rough case. i.e. replace W by $\mathbf{Y} \in \mathcal{C}^\alpha$.

④ Pathwise Stoch. optimal control.:

Consider $dY_t = b(t, Y_t, \eta_t)dt + \sigma(t, Y_t, \eta_t)$

$$dB_t + f(Y_t)dW_t.$$

We want to figure out $V(t, y, w) = \inf_{h} \mathbb{E}^{t,y}(g(Y_T) \mid \mathcal{F}_T^W)(w)$

Let W fixed. then we have dynamic programming principle: $V(t,y) = \inf_{h} \mathbb{E}^{t,y}(V(t+h, Y_{t+h}))$

<u>Rmk</u>: Remove "inf". it satisfies Kolmogorov backward equation.

By expanding at (t, Y_t). i.e. $\mathbb{E}^{t,y}(V(t+h, Y_{t+h})) = V(t,y) + (\partial_t + \mathcal{L}_t + \Gamma W) V(t,y) h$

We also have stochastic HJB eq.:

$-dV_t = F(D^2V_t, DV_t, y, t)dt + f(y) \cdot DV_t \, dW_t$.

$V_T = g$ where $F(DV^2, DV, y, t) = \inf_{h} \mathcal{L}_t V$

<u>Rmk</u>: Replace W by rough path. then RSDE gets in.