Sto-Rough Integral

(1) <u>Rough Itô process:</u>

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $\mathcal{F}_t$-BM B_t.

<u>Denote:</u> For $X : \mathbb{R}^{\geq 0} \to \mathbb{R}^1$. $A : \Delta \subset \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}^1$.

$\delta X_{st} := X_t - X_s$. $\delta A_{sut} := A_{su} + A_{ut} - A_{st}$

<u>Def:</u> Rough Itô process w.r.t. $\mathbf{Y} = (Y, \mathbb{Y}) \in \mathscr{C}^\alpha$ is $X_t = X_0 + \int_0^t (F_s, F_s')(\omega)\, d\mathbf{Y}_s + \sigma_s(\omega)\, dB_s + b_s(\omega)\, ds$. where b, σ and (F_s, F_s') are progressive. SRI $\int_0^t (F_s, F_s')\, d\mathbf{Y}_s$ is assumed to exist in sense of:

$$\int_0^\cdot (F, F')\, d\mathbf{Y} := \mathbb{P}\text{-}\lim_{|\pi| \to 0} \sum F_s\, \delta Y_{st} + F_s' \mathbb{Y}_{st}$$

<u>Rmk:</u> For $\alpha > 1/2$, $(F, F') = (M, 0)$ for some $M \in \mathcal{M}^{loc.c}$. The SRI can be also well-def (since $M \in V^{2+\varepsilon}$ by Lépingle $\Rightarrow$ Young integral). So it also works for semimart. $X = M + V$.

<u>Lem.</u> For $\sigma \in L^\infty[0,T]$. a.s. $M_t = \int_0^t \sigma_s(\omega)\, dB_s \in$

C^α. a.s. $\forall \alpha < 1/2$. And $(M, \int M \, dM) \in \mathscr{C}^\alpha$. a.s. $\forall \alpha < 1/2$.

Pf: By DDS represent: $M_t = B_{\langle M \rangle t}$

And $\langle M \rangle = \int \sigma_s^2 \, ds \in \text{Lip}$.

Rmk: For $\vec{B}$ is d-dim. Note $(M_t^1, \cdots M_t^d)$ will not have represent $\vec{B}_{\langle M \rangle}$. But we can still apply Kolmogorov criteria for RP on it.

Denote: $A \overset{\gamma}{=} \bar{A}$ if $|A_{s,t} - \bar{A}_{s,t}| \lesssim |t-s|^\gamma$ on $[0,T]$ uniformly.

e.g. $(Y, \mathbb{Y}) \in \mathscr{C}^\alpha \Rightarrow Y \overset{\alpha}{=} 0$. $\mathbb{Y} \overset{2\alpha}{=} 0$.

Lem. (Sewing)

$(\Xi_{st})_{[0,T]}$ satisfies $(\delta \Xi)_{sut} \overset{1+\varepsilon}{=} 0$. Then $\exists$ unique conti. path $I(t)$ s.t. $I(0) = 0$. $|\delta I - \Xi_{\cdot\cdot}| \overset{1+\varepsilon}{=} 0$. $\delta I_{s,t} = \lim_{|\mathcal{P}| \to 0} \sum \Xi_{u,v}$

Def: $f \in \text{Lip}^\gamma(\mathbb{R}^n, \mathbb{R}^e)$ if $f \in C^{\lfloor \gamma \rfloor}$ and $D^{\lfloor \gamma \rfloor} f \in \text{Lip}^{\{\gamma\}}$. $D^k f \in \text{Lip}^1$. $\forall k = \lfloor \gamma \rfloor - 1$

Rmk: Recall controlled RP is stable under Lip^2.

Next, we define sol. for

$dX_t = f_t(X_t)\,dY_t$ in sense of

$$X_t = X_0 + \int_0^t (f_s(X_s),\ ((Df_s)f_s + f_s')(X_s))\,d\mathbf{Y}_s.$$

Rmk: The controlled term is just consider locally: $f_s'(X_s)$ is s-derivative and $Df_s(X_s)f_s(X_s) \approx Df_s(X_s)X_s'$ by RDE.

e.g. Let $f_s(X_s) = A_s X_s$. A_s is a linear transf. So the controll part of RI is $(A_s X_s,\ (A_s^2 + A_s')X_s)$

Then the sol. flow is linear operator to itself: $X_t = \pi^{\mathbf{Y}}_{t\leftarrow 0} X_0$. $\pi^{\mathbf{Y}}_{t\leftarrow 0} \in \mathcal{L}(\mathbb{R}^d)$.

Rmk: For $dX = (A, A')X\,d\mathbf{Y}$ in $d=1$.

$$\pi^{\mathbf{Y}}_{t\leftarrow 0} = \exp\Big(\int_0^t (A_s, A_s')\,d\mathbf{Y}_s - \frac{1}{2}\int_0^t A_s^2\,d[\mathbf{Y}]_s\Big)$$

(2) **RSDEs Preview**:

Consider $(f_{s,\omega}, f'_{s,\omega})$, $\sigma_{s,\omega}$, $b_{s,\omega}$ are progressive. And RSDE defined by:

$$dX_s = (f_{s,\omega}, f'_{s,\omega})(X_s)\,dY_s + \sigma_{s,\omega}(X_s)\,dB_s + b_{s,\omega}(X_s)\,ds$$

its sol. is defined by:

$$X_t = X_0 + \int_0^t (f_{s,\omega}, f'_{s,\omega})(X_s)\,dY_s + \sigma_{s,\omega}(X_s)\,dB_s + b_{s,\omega}(X_s)\,ds$$

where $\int RI = \mathbb{P}\text{-}\lim \sum \{ f_{s,\omega}(X_s)\,\delta Y_{s,t} +$

$$(f_{s,\omega} Df_{s,\omega} + f'_{s,\omega})(X_s)\,\mathbb{Y}_{s,t} \}.$$

e.g. i) (Path-indept.)

For $(f_{s,\omega}, f'_{s,\omega}) = (F, F') \in \mathcal{D}_Y^{2\gamma}$, a.s.

Let $Z_t \stackrel{\Delta}{=} X_0 + \int_0^t (F, F')\,dY_s$

$\Rightarrow X_t = Z_t + \int_0^t \sigma_{s,\omega}(X_s)\,dB_s + b_{s,\omega}(X_s)\,ds$

i.e. It's reduced to SDE case.

ii) (linear deterministic.)

For $(f_{s,\omega}, f'_{s,\omega})(x) = (A_s, A'_s)(x) \in \mathcal{D}_Y^{2\gamma}$.

Its sol. satisfies SVE:

$$X_t \stackrel{(*)}{=} \Pi^{Y}_{t\leftarrow 0} X_0 + \int_0^t \Pi^{Y}_{t\leftarrow s} (\sigma_{s,\omega}(X_s)\,dB_s +$$
$$b_{s,\omega}(X_s)\,ds)$$

<u>Rmk</u>: If $(A_s, A'_s) \stackrel{a.s.}{\in} \mathcal{D}_Y^{2\gamma}$ & progressive

Then $\Pi^{Y}_{t\leftarrow s} \in \mathcal{F}_t$. And it's not easy to write $\Pi^{Y}_{t\leftarrow s}(\square\, dB_s + \square\, ds)$

$$= \square\, dB_s + \square\, ds \text{ i.e.}$$

Itô formulation. ($\Pi^{Y}_{t\leftarrow s}$ involve random)

Set $\widetilde{X}_t = \Pi^{Y}_{s\leftarrow t} X_t$. $\Rightarrow$ it transfers RSDE (*) to SDE:

$$\widetilde{X}_t = X_0 + \int_0^t \Pi^{Y}_{0\leftarrow s}\{\sigma(s,\omega)(\Pi^{Y}_{0\leftarrow s}\widetilde{X}_s)\,dB_s$$

$$+ b(s,\omega)(\Pi^{Y}_{0\leftarrow s}\widetilde{X}_s)\,ds\}$$

$\Rightarrow$ We solve RSDE (*) by recovering X_t from $\widetilde{X}_t$: $X_t = \Pi^{Y}_{t\leftarrow 0}\widetilde{X}_t$ (It's called flow transformation technique.

iii) (Joint Rough Path approach)

For $X_t = X_0 + \int_0^t f(X_s)\,dY_s + \int_0^t \sigma(X_s)\,dB_s$

$$+ \int_0^t b(X_s)\,ds.$$

Although rough integral and Itô's integral are different integration system. They're still consistent in the case of BM.

So we can consider joint rough lift

$(\mathbf{Y}, B) =: J$

$\Rightarrow dX_t = X_0 + \int_0^t g(X_s)\, dJ_s + \int_0^t b(X_s)\, ds$

where $g = (f, \sigma)$. become stoch. RDE.

(3) <u>Stochastic Rough integral</u>:

① <u>Stochastic Sewing</u>:

<u>Denote</u>: i) $\mathbb{E}_s(\cdot) := \mathbb{E}(\cdot \mid \mathcal{F}_s)$. For $A: \Delta \to \mathbb{R}^1$, set

$\|A_{st}\|_{m,n} := \| \|A_{st} \mid \mathcal{F}_s\|_m \|_n$

<u>Rmk</u>: $\|A_{st}\|_{m,m} = \|A_{st}\|_m$.

$A_{st} \in \mathcal{F}_s \Rightarrow \|A_{st}\|_{m,n} = \|A_{st}\|_m$.

ii) $(\mathbb{E}.(A))_{st} = \mathbb{E}_s(A_{st})$.

$(\mathbb{E}.(B))_{sut} = \mathbb{E}_s(B_{sut})$

<u>Lem</u>. (Doob decomp.)

$(X_k)_0^N \subset L^1(\Omega; \mathbb{R}^d)$ adapted to $\mathcal{F}_k \Rightarrow$

There exists unique $(y_k), (Z_k)$ s.t. y_k is $\mathbb{R}^d$-mart. with $y_0 = X_0$ and $Z_k :=$ $\sum_{j=1}^k \mathbb{E}_{j-1}(\delta X_j) \in \mathcal{F}_{k-1}$ pred.

Lem. (Condition BDG)

$(X_k)_0^N$ is d-dim L^m, $\mathcal{G}_k$-mart. s.t. $1 \le m < \infty$. For $G \in \mathcal{G}_0$, we have:

$\| S x_N \mid G \|_m \overset{i)}{\lesssim} \| M x_N \mid G \|_m \overset{ii)}{\lesssim} \| S x_N \mid G \|_m$ for

$S x_N := (|x_0|^2 + \sum_1^N |\delta x_k|^2)^{\frac{1}{2}}$. $M x_N = \max_{0 \le k \le N} |x_k|$

Pf: i) is trivial. As for ii): we only consider $m=1$. i.e. Davis inequi.

For $d=1$: Note we have: $\forall (f_k) \subset \mathbb{R}$.

$M f_N \le 6 S f_N + 2 \sum_0^{N-1} h_k (f_{k+1} - f_k)$ where

$h_k := f_k / [(S f_k)^2 + (M f_k)^2]^{1/2}$.

Rmk: It's kind of superhedging.

Let $f_k = X_k$ since the second term of RHS is martingale transform.

Take $\mathbb{E}(\cdot \mid G)$ on both side $\Rightarrow$ ii)

To obtain $d > 1$ case:

Note: $M f_k \le \sum_1^d M f_k^i$. $\sum_1^d S f_k^i \overset{\text{Cauchy}}{\le} \sqrt{d}\, S f_k$.

Rmk: To extend to $m > 1$. We can use Garsia-Neveu Lem.:

$$u^p = p(p-1)\int_0^\infty \lambda^{p-2}(u-\lambda)_+ d\lambda.$$

to reduce $m>1$ to $m=1$ (Davis)

Thm. (Stochastic sewing in mixed moment)

For $2 \le m \le n \le \infty$. $M < \infty$. $A_{\cdot\cdot}(\omega): \Omega \times \Delta \to \mathbb{R}^{d_A}$ is sto. two para. func. s.t. $A_{ss} = 0$. $A_{st} \in \mathcal{F}_t$ $\forall (s,t) \in \Delta$. If uniformly on $[0,T]$. We have:

$\| \mathbb{E}_\cdot (\delta A_{\cdot,\cdot,\cdot}) \|_n \overset{1+\varepsilon}{=} 0$. $\| \delta A_{\cdot,\cdot,\cdot} \|_{m,n} \overset{\frac{1+\varepsilon}{2}}{=} 0$.

Then for $\forall$ partition $\pi = (t_i)$ of $[s,t]$.

$A^\pi_{st} := \sum_0^{n-1} A_{t_i, t_{i+1}}$ $\quad J^\pi_{st} := A^\pi_{st} - A_{st}$.

i) J^π_{st} is L^m - Cauchy $\xrightarrow{L^m} J_{st} \in L^m$

ii) $J_{st} = (\delta \mathcal{A})_{st} - A_{st}$. where $\mathcal{A}_t :=$ $\lim_{|\pi| \to 0} A^\pi_{0t}$ in pr. is $\mathcal{F}_t$ - adapted.

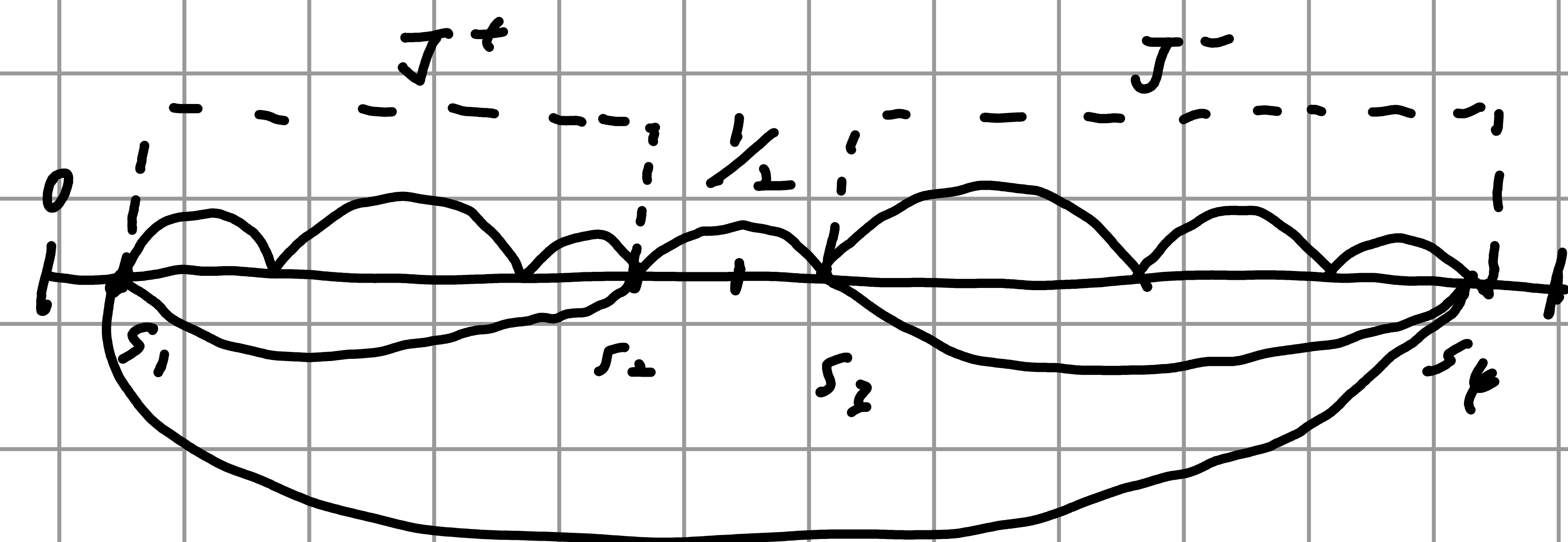
iii) $\| \mathbb{E}_\cdot (\delta \mathcal{A} - A) \|_n \overset{1+\varepsilon}{=} 0$ $\quad \| \delta \mathcal{A} - A \|_{m,n} \overset{\frac{1+\varepsilon}{2}}{=} 0$.

Lem. (Dyadic Allocation)

Set $k_i^n := s + \frac{i}{2^n}(t-s)$. We have: $J^\pi_{st} = \sum_{n \ge 0} \sum_{i=0}^{2^n - 1} R_i^n$. where $R_i^n = 0$. $\forall i$ for $n \ge \pi(n)$

for some $\pi(n)$. and $\exists (s_i)_1^4 \subset \pi \cap [d_n^i, d_n^{i+1}]$. s.t. $\hat{R_i} = \delta A_{s_1 s_2 s_3} + \delta A_{s_1 s_3 s_4}$

<u>Pf</u>: WLOG. $[s, t] = [0, 1]$. For $n = 0$:



Assume the upper arcs $= \sum A_{t_i t_{i+1}}$

there's one arc containing $1/2$ with arcs in $J^+_{s_1 s_2} \subset [0, \frac{1}{2}]$ and arcs in $J^-_{s_3 s_4} \subset [\frac{1}{2}, 1]$ in $\sum A_{t_i t_{i+1}} = A_{01}^{\pi}$. So:

$$J_{s_1 s_4} + A_{s_1 s_4} = A_{01}^{\pi} = (J^+_{s_1 s_2} + A_{s_1 s_2}) + A_{s_2 s_3} + (J^-_{s_3 s_4} + A_{s_3 s_4})$$

$$\Rightarrow J_{s_1 s_4} = J^+_{s_1 s_2} + J^-_{s_3 s_4} + R.$$

where $R = \delta A_{s_1 s_2 s_3} + \delta A_{s_1 s_3 s_4} = R_0^0$

$\Rightarrow$ We can recursively repeat it inside $J^+ \subset [0, \frac{1}{2}]$ and $J^- \subset [\frac{1}{2}, 1]$.

<u>Lem</u>$^+$. (Doob/BDG estimate for random sum)

For $\infty > m \geq 2$. $\infty \geq p \geq 2$. $(y_k)_{k \geq 0} \subset L^m$

is $\mathcal{F}_k$-adapted. Fix $G \subset \mathcal{F}_0$. Denote

$\|\cdot\|_{m,n,G} := \| \| \cdot 1_G \|_m \|_n$. Then:

$$\Big\| \sum_0^{N} \eta_k \Big\|_{m,n,G} \lesssim \sum_1^{N} \| \mathbb{E}_{k-1}(\eta_k) \|_{m,n,G} + \Big(\sum_0^{N} \|\eta_k\|_{m,n,G}^2 \Big)^{\frac{1}{2}}$$

<u>Pf</u>: Set $f_k = \eta_0 + \sum_1^{k} \eta_i - \mathbb{E}_{i-1}(\eta_i)$

$\Rightarrow f_k$ is L^m-mart w.r.t. $\mathcal{F}_k$.

$$LHS \lesssim \Big\| \sum_1^{N} \mathbb{E}_{k-1}(\eta_k) \Big\|_{m,n,G} + \| f_N \|_{m,n,G}$$

$$\overset{\text{cond. BDG}}{\lesssim} \sum \| \mathbb{E}_{k-1}(\eta_k) \|_{m,n,G} + \Big\| \Big(\sum_0^{N} |\delta f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{m,n,G}$$

$$\overset{\text{minkov.}}{\lesssim} \square + \Big(\sum \| |\delta f_k|^2 \|_{\frac{m}{2}, \frac{n}{2}, G} \Big)^{\frac{1}{2}}$$

With $\| |\delta f_k|^2 \|_{\frac{m}{2}, \frac{n}{2}, G} = \| \delta f_k \|_{m,n,G}^2$

$\leq 2 \|\eta_k\|_{m,n,G}$.

<u>Lem.</u>[5] (Uniform estimate)

Under assumpt of Sto-sew. We have:

$\| \mathbb{E}_\cdot (J^{\pi}_{\cdot\cdot}) \|_n \overset{1+\varepsilon}{=} 0$. $\| J^{\pi}_{\cdot\cdot} \|_{m,n} \overset{\frac{1+\varepsilon}{2}}{=} 0$.

<u>Pf</u>: $J^{\pi} = \sum_{N \geq 0} \sum_{i=0}^{2^N - 1} R_i = \sum \sum (\delta A_{S_1^{N,i} S_2^{N,i} S_3^{N,i}} + \delta A_{\square})$

And $\max_{j_1, j_2} | S_{j_1}^{N,i} - S_{j_2}^{N,i} | \leq |t-s| / 2^N$.

Let $\mathcal{G}_i^N := \mathcal{F}_{S_1^{N,i}} \Rightarrow R_i^N \in \mathcal{G}_{i+1}^N$

By assumpt of Sto-sew:

$$\|\mathbb{E}(R_i^N|\mathcal{G}_i^N)\|_n \lesssim \left(\frac{|t-s|}{2^N}\right)^{1+\varepsilon}$$

$$\|R_i^N\|_{n,m} \lesssim \left(\frac{|t-s|}{2^N}\right)^{\frac{1+\varepsilon}{2}}$$

And $\|\sum_0^{2^N-1} R_i^N\|_{m,n,\mathcal{F}_{t_0}} \lesssim \sum_0^{2^N-1} \|\mathbb{E}(R_i^N|\mathcal{G}_i^N)\|_{m,n,\mathcal{F}_{t_0}}$

$$+ \left(\sum \|R_i^N\|^2_{m,n,\mathcal{F}_{t_0}}\right)^{\frac{1}{2}}$$

$$\lesssim \sum \|\mathbb{E}(R_i^N|\mathcal{G}_i^N)\|_n + \left(\sum \|R_i^N\|^2_{m,n,\mathcal{G}_i^N}\right)^{\frac{1}{2}}$$

from Jensen inequi. and contraction property of conditional expectation.

So: $\|\sum_0^{2^N-1} R_i^N\|_{m,n,\mathcal{F}_{t_0}} \lesssim \sum |\lambda_{i+1}^N - \lambda_i^N|^{1+\varepsilon} + \left(\sum |\lambda_{i+1}^N - \lambda_i^N|^{1+\varepsilon}\right)^{\frac{1}{2}}.$

$$\lesssim |t-s|^{\frac{1+\varepsilon}{2}} 2^{-N\varepsilon/2}.$$

$\Rightarrow \|J^\pi\|_{m,n,\mathcal{F}_{t_0}} \overset{\frac{1+\varepsilon}{2}}{=} 0.$

And $\|\mathbb{E}_{t_0}(J^\pi)\|_n \le \sum\sum \|\mathbb{E}(R_i^N|\mathcal{F}_{t_0})\|_n$

$$\overset{\text{contract}}{\le} \sum\sum \|\mathbb{E}(R_i^N|\mathcal{G}_i^N)\|_n \lesssim |t-s|^{1+\varepsilon}$$

Next, we can return proof of Sto-Sew:

i) For $\pi = (t_i)_0^N$, $\pi' = (s_i)_0^{N'}$ partitions of $[0,t]$. Set $\pi'' = \pi \cup \pi' = (u_i)_0^{N''}$. $N'' \le N + N'$.

$\Rightarrow A^{\pi}_{0t} - A^{\pi'}_{0t} = \sum_{i=0}^{n-1} Z_i$. $Z_i = \sum_{j: t_i \le u_j < t_{i+1}} A_{u_j u_{j+1}} - A_{t_i t_{i+1}}$.

By Doob/BDG estimate:

$$\| A^{\pi''}_{0t} - A^{\pi}_{0t} \|_m \lesssim \sum_0^{n-1} \| \mathbb{E}_{t_i}(Z_i) \|_m + \left(\sum_0^{n-1} \| Z_i \|_m^2 \right)^{\frac{1}{2}}$$

$$\overset{\text{Lem 5}}{\lesssim} \sum \Delta t_i^{1+\varepsilon} + \left(\sum \Delta t_i^{1+\varepsilon} \right)^{\frac{1}{2}}$$

$$\lesssim t^{\frac{1}{2}} |\pi|^{\varepsilon/2} \vee t |\pi|^{\varepsilon}$$

Similarly, we have: $\| A^{\pi}_{0t} - A^{\pi'}_{0t} \|_m \lesssim |\pi|^{\frac{\varepsilon}{2}} \vee |\pi'|^{\frac{\varepsilon}{2}}$

So: $J^{\pi}_{0t} = A^{\pi}_{0t} - A_{0t}$ is L^m-Cauchy $\to J_{0t}$.

ii) Since $A_0 := \lim A^{\pi}_{0t} = J_{0t} + A_{0t}$ from i).

Repeat construct on $[0,s]$. $[s,t]$.

(Note it converges on arbitrary mesh, we

can set $\pi[0,t] = \pi'[0,s] \cup \pi''[s,t]$. $|\pi| \to 0$)

$J_{st} + A_{st} = \lim A^{\pi''}_{st} = \lim A^{\pi}_{0t} - A^{\pi'}_{0s} = A_t - A_s$

iii) Using Fatou's Lem ($\underline{\lim}$ case) and J^{π}_{st}

$\xrightarrow{L^m} J_{st} = \delta A_{st} - A_{st} \Rightarrow \exists \pi'$ subseq of π. st.

$\mathbb{E}(J^{\pi'}_{st} | \mathcal{F}_s) \xrightarrow{a.s.} \mathbb{E}(J_{st} | \mathcal{F}_s)$. Apply Lem 5:

So $0 = \underline{\lim} |\pi'|^{1+\varepsilon} \geq \underline{\lim} \| \mathbb{E}.(A^{\pi'} - A) \|_n$

$$\geq \| \underline{\lim}\, \mathbb{E}.(A^{\pi'} - A) \|_n = \square$$

Also since $\|A - A^{\pi}\|_m \to 0$. So $\|A - A^{\pi} | \mathcal{F}_\cdot\|_m \to 0$.

Then apply Fatou's to get another one.

Rmk: i) It can also work for Banach space with mart. type p.

ii) $m = \infty$ is a strong assumption so we drop it. Since BDG doesn't work in L^{∞} ($m=\infty$ even excludes BM case)

② Stochastic Control Rough path:

Def: $(A_{st}(w))_{s t \in \Delta} \in C_2^{\alpha} L_{m,n}$ if $\|A\|_{\alpha,m,n} := \sup\limits_{0 \le s,t \le T} \| \|A_{st} | \mathcal{F}_s\|_m \|_n / |t-s|^{\alpha} < \infty$ and we denote it by $A \overset{\alpha,m,n}{=} 0$.

Rmk: For path $(X_t(w)) \in C^{\alpha} L_{m,n}$ if $\delta X \in C_2^{\alpha} L_{m,n}$. Denote:

$$[[X]]_{\alpha,m,n} \triangleq \|\delta X\|_{\alpha,m,n}$$

$$\|X\|_{\alpha,m,n} := [[X]]_{\alpha,m,n} + \sup_{[0,T]} \|X_t\|_m$$

prop. For $1 \le m \le n \le \infty$ and $\alpha > 0$. Then:

i) $C_2^{\alpha} L_{m,n}$ is Banach for $\forall \alpha > 0$.

ii) $C_2^{\alpha} L_{m,n} \hookrightarrow C_2^{\alpha} L_{m',n'}$. $\forall m \ge m', n \ge n'$.

iii) If $A^k \to A$ in $C_2^{\kappa} L_m$. Then:

$$\overline{\lim_k} \|A^k\|_{\kappa,m,\infty} \geq \|A\|_{\kappa,m,\infty}$$

Thm. (John - Nirenburg)

X is conti. $\mathcal{F}_t$-adapted. s.t. $\delta X \in C_2^{\kappa} L_{1,\infty}$

$\Rightarrow \forall \lambda > 0.\ \log \mathbb{E}(e^{\lambda \sup_{0 \leq s \leq t \leq T} |\delta X_{st}|}) \leq CT(\lambda \|\delta X\|_{\kappa,1,\infty})^{\frac{1}{\kappa}}$

where C is indept of λ, Y, T.

Def: $(Z, Z') \in D_Y^{2\alpha} L_{m,n}$ is Ito-controlled RP

if i) (Z, Z') is progressive.

ii) $\delta Z \in C_2^{\alpha} L_{m,n}$

iii) $\mathbb{E}_{\cdot}(R^Z_{\cdot\cdot}) \in C_2^{2\alpha} L_n$

iv) $Z' \in C^{\alpha} L_{m,n}$. $\sup_t \|Z'_t\|_n < \infty$.

Denote $\|(Z, Z')\|_{D_Y^{2\alpha} L_{mn}} = \|\delta Z\|_{\alpha,m,n} + \|\delta Z'\|_{\alpha,m,n}$

$+ \sup_t \|Z'_t\|_n + \|\mathbb{E}_{\cdot}(R^Z)\|_{2\alpha,n}$ seminorm on it.

Rmk: $(Z, Z') \in D_Y^{2\alpha} L_{m,n}$ can imply:

i) $\mathbb{E}_{\cdot}(R^Z) \overset{2\alpha,n}{=} 0$. ii) $\mathbb{E}_{\cdot}(\delta Z') \overset{\alpha,n}{=} 0$.

iii) $R^Z \overset{\alpha,m,n}{=} 0$. iv) $Z' \overset{\alpha,n}{=} 0$.

Note for iii): $(n > m)$

$\|R^Z\|_{m,n,\alpha} \le \sup_t \|Z'_t\|_n |\delta Y|_\alpha + \|\delta Z'\|_{m,n,\alpha}$

Lem. Def iv) $\Rightarrow$ Rmk ii)

Def iii) $\Rightarrow$ Rmk i)

Pf: By Hölder and Jensen inequ.

Thm. If $(Z, Z') \in D_Y^{2\alpha} L_{m,n}$, $Y \in C^\alpha$. $\alpha \in (\frac{1}{3}, \frac{1}{2}]$

$2 \le m \le n \le \infty$. $m < \infty$. Then: SRI exists

$I_t := \int_0^t (Z, Z')\, dY = \lim_{|\pi| \to 0} I(Z\delta Y + Z' \mathbb{Y})$

Pf: Apply Sto-Sew Thm on $A :=$

$Z\delta Y + Z'\mathbb{Y} \Rightarrow \delta A = R^Z \delta Y + \delta Z' \mathbb{Y}$

(It also works for $\varepsilon_1, \varepsilon_2 > 0$ in

$1+\varepsilon_1$ and $1+\varepsilon_2/2$, $3\alpha = 1+\varepsilon_1$, $2\alpha = 1+\varepsilon_2/2$)

Combined with Rmk above.

Thm. $(Z, Z') \in D_Y^{2\alpha} L_{m,n}$. If $\|Z\|_{n,\infty} < \infty$. Then:

$(\int Z\, dY, Z) \in D_Y^{2\alpha} L_{m,n}$ with estimate

$\|(\int Z\, dY, Z)\|_{D_Y^{2\alpha} L_{m,n}}$

$\le C_\alpha \|(Z, Z')\|_{D_Y^{2\alpha} L_{m,n}} + \|Z\|_{n,\infty}$

Pf: $\|(\int Z\, dY, Z)\|_{D_Y^{2\alpha} L_{m,n}} = \sup_t \|Z_t\|_n + \|\delta Z\|_{\alpha,m,n}$

$\|\mathbb{E}.(\int Z\, dY - Z\delta Y)\|_{2\alpha,n} + \|\int Z\, dY\|_{\alpha,m,n}$

Note $\|\mathbb{E}.(\int Z\,dY - Z\delta Y - Z'\mathbb{Y})\|_n \overset{3\gamma}{=} 0.$

$\|\int Z\,dY - Z\delta Y - Z'\mathbb{Y}\|_{m,n} \overset{2\gamma}{=} 0.$

(Const. consists of $\|\square\|_{\gamma,m,n}$, $\|\square\|_{2\gamma,n}\cdots$)

So: RHS $\leq \|Z\|_{n,\infty} + \|\mathbb{E}.(Z'\mathbb{Y})\|_{2\gamma,n} +$

$\|Z\delta Y + Z'\mathbb{Y}\|_{\gamma,m,n} + \|\delta Z\|_{\gamma,m,n}$

$\leq \|Z\|_{n,\infty} + \|Z'\|_{n,\infty}\|\mathbb{Y}\|_{2\gamma}(1+T^{\gamma}) +$

$T^{\gamma}\|R^Z\|_{2\gamma,m,n} + \|\delta Z\|_{\gamma,m,n} + \|\delta Z\|_{\gamma,m,n}$

With $\|R^Z\|_{2\gamma,m,n} \overset{(n\geq m)}{\leq} \|\mathbb{E}.(R^Z)\|_{2\gamma,n}$

__Cor.__ (Stability)

For $(\bar{Z},\bar{Z}') \in D^{2\gamma}_{\bar{Y}}L_{m,n}$, $\bar{\mathbb{Y}} \in \mathscr{C}^{\gamma}$ and $\|\bar{Z}\|_{n,\infty} < \infty$. Then:

$\|(\int Z\,d\mathbb{Y}, Z; \int \bar{Z}\,d\bar{\mathbb{Y}}, \bar{Z})\|_{\mathbb{Y},\bar{\mathbb{Y}},2\gamma,m,n} \lesssim$

$\|\delta(Y-\bar{Y})\|_{\gamma} + \|\delta(\mathbb{Y}-\bar{\mathbb{Y}})\|_{2\gamma} + \|Z-\bar{Z}\|_{n,\infty} +$

$\|\delta(Z-\bar{Z})\|_{\gamma,m,n} + T^{\gamma}\|Z,Z';\bar{Z},\bar{Z}'\|_{\mathbb{Y},\bar{\mathbb{Y}},2\gamma,m,n}$

③ Composition with Regular func.:

For $(X,X') \in D^{2\gamma}_{Y}L_{mn}$, $g \in Lip^2$. We consider the composition $(Z,Z') := (g(X), Dg(X)X')$

__Rmk:__ i) If g is linear, obviously $(Z,Z') \in$

$D_Y^{2\tau} L_{m,n}$ still.

ii) If $g = g(t,x)$. We will consider Z'
$= g'(t,X) + Dg(t,X)X'$.

$\Rightarrow R^Z_{st} = g_t(X_t) - g_s(X_t) - g'_s(X_s)\delta Y_{st}$
$\qquad + g_s(X_t) - g_s(X_s) - Dg_s(X_s)X'_s\delta Y_{st}$

The following prop. also holds for $(g_t, g'_t) \in D_Y^{\beta,\beta'} L_{m,n} C_b^\gamma$. sto. contro. vec. field.

<u>prop.</u> For $g \in Lip^2$. $(X,X') \in D_Y^{2\tau} L_{m,n}$. $2 \le m \le n$ for $n < \infty$. Then: $(Z,Z') = (g(X), Dg(X)X')$ $\in D_Y^{2\tau} L_{m,\frac{n}{2}}$. Besides.

$\|(Z,Z')\|_{D_Y^{2\tau} L_{m,\frac{n}{2}}} \le C_{\|g\|_{Lip^2}} (1 + \|(X,X')\|_{D_Y^{2\tau} L_{m,n}})$

<u>Rmk</u>: Note that n is half. It's the reason that we consider $D_Y L_{m,n}$ space in the following.

<u>Pf</u>: $\|\delta Z\|_{\tau,m,n}$. $\|\delta Z'\|_{\tau,m,n}$. $\|Z'\|_{0,n}$ are bdd from $X \in D_Y^{2\tau} L_{m,n}$. $g \in Lip^2$.

As for $R^Z = \delta g(X) - Dg(X)X'\delta Y$
$= \delta g(X) - Dg(X)\delta X + Dg(X)R^X$

Let $Lg(X_s, X_t) := g(X_t) - g(X_s) - Dg(X_s)\delta X_{st}$

$= [\int_0^1 (Dg(X_s + \theta\delta X_{st}) - Dg(X_s) d\theta] \delta X_{st}$

$\leq C_g |\delta X_{s,t}|^2$ ($\Rightarrow$ cause halfing of n)

$\| \mathbb{E}_\cdot |\delta X_{st}|^2 \|_{\frac{n}{2}} = \|\delta X_{st}\|_{2,n}^2 \overset{X\in\square}{\lesssim} |t-s|^{2\tau}$

$\| Dg(X_s) \mathbb{E}_s (R^X_{st}) \|_{\frac{n}{2}} \leq |Dg|_\infty \| \mathbb{E}_\cdot (R^X) \|_n$

$\lesssim |t-s|^{2\tau}$

$\Rightarrow \| \mathbb{E}_\cdot (R^Z) \|_{2\tau, \frac{n}{2}} \leq C_{\|g\|_{Lip^2}} (\|\delta X\|^2_{\tau, m, n} +$

$\| \mathbb{E}_\cdot (R^X) \|_{2\tau, n})$

<u>prop.</u> (Stability of composition)

Let $g \in Lip^2$. (X, X'). $(\bar{X}, \bar{X}')$ bounded in $D_\gamma^{2\tau} L_{m\infty}$. $2 \leq m < \infty$. Then:

$\|(Z, Z') - (\bar{Z}, \bar{Z}')\|_{\gamma, 2\tau, m} \lesssim \| |X_0 - \bar{X}_0| \wedge 1 \|_m$

$+ C_R \| (X, X') - (\bar{X}, \bar{X}') \|_{\gamma, 2\tau, m}$

where $\|(X, X')\|_{\gamma, 2\tau, m, \infty} \vee \|(\bar{X}, \bar{X}')\|_{\gamma, 2\tau, m, \infty} \leq R$.

<u>Pf</u>: Similarly as above. We focus on $R^\square$:

$R^Z - R^{\bar{Z}} = Lg(X_s, X_t) - Lg(\bar{X}_s, \bar{X}_t) +$

$Dg(X_s)\delta X_{st} - Dg(\bar{X}_s)\delta \bar{X}_{st}$.

Set $X^\theta = \theta X + (1-\theta)\bar{X}$. $\tilde{X} = X - \bar{X}$.

$\Rightarrow L g(X_s, X_t) - L g(\bar{X}_s, \bar{X}_t)$

$= \int_0^1 \frac{d}{d\theta}(L g(X_s^\theta, X_t^\theta))\, d\theta$

$= \int_0^1 (Dg(X_t^\theta)\tilde{X}_t - Dg(X_s^\theta)\tilde{X}_s - Dg(X_s^\theta)\delta\tilde{X}_{st}$

$\qquad - D^2 g(X_s^\theta)\langle \tilde{X}_s, \delta X_{st}^\theta\rangle)\, d\theta$

$= \int_0^1 A + B\, d\theta$

$|A| = |Dg(X_t^\theta)\tilde{X}_s - Dg(X_s^\theta)\tilde{X}_s$

$\qquad - D^2 g(X_s^\theta)\langle \tilde{X}_s, \delta X_{st}^\theta\rangle|$

$\leq |\tilde{X}_s|\, |g|_{Lip^2}\, |\delta X_{st}^\theta|^2$

$|B| = |Dg(X_t^\theta)\delta\tilde{X}_{st} - Dg(X_s^\theta)\delta\tilde{X}_{st}|$

$\qquad \leq |g|_{Lip^2}\, |\delta X_{st}^\theta|\, |\delta\tilde{X}_{st}|$

$|\delta X_{st}^\theta| \leq |\delta X_{st}| + |\delta\bar{X}_{st}| =: m_{st}$

So $|L g(X_s, X_t) - L g(\bar{X}_s, \bar{X}_t)| \leq m_{st}^2 |\tilde{X}_s| +$

$\qquad m_{st}\, |\delta\tilde{X}_{st}|$

With $|L g(X_s, X_t) - L g(\bar{X}_s, \bar{X}_t)|$

$\leq |\delta X_{st}|^2 + |\delta\bar{X}_{st}|^2 \lesssim m_{st}^2$ from above.

$\Rightarrow \mathbb{E}_s(|L g(X_s, X_t) - L g(\bar{X}_s, \bar{X}_t)|) \leq$

$(|\tilde{X}_s| \wedge 1)\, \mathbb{E}_s(m_{st}^2) + \mathbb{E}_s(m_{st}\, |\delta\tilde{X}_{st}|)$

Since $\mathbb{E}_s(m_{st}^2) \leq \|m\|_{2,\infty}^2 \leq \|m\|_{\tau,2,\infty}^2 |t-s|^{2\tau}$.

with $\|\tilde{X}_s\|_n \leq \|\tilde{X}_0\|_n + \|\delta\tilde{X}\|_{n,\tau} T^\tau$

Lem. $\|AB\|_{p,q} \le \|A\|_{p',q'} \cdot \|B\|_{p'',q''}$ for $1/p \le 1/p' + 1/p''$ and $1/q \le 1/q' + 1/q''$

Pf: use Hölder inequi. twice.

$\Rightarrow \|E_s(M_{st}|\delta \tilde{X}_{st}|)\|_n \le \|M_{st}|\delta \tilde{X}_{st}|\|_n$

$\overset{\text{Lem.}}{\le} \|M_{st}\|_{2,\infty} \|\delta \tilde{X}_{st}\|_{n,n}$

$\le \|M\|_{2,\infty,\gamma} \|\delta \tilde{X}_{st}\|_{n,\alpha} |t-s|^{2\gamma}$.

B can also be estimated by above.

By the cond. $\Rightarrow n = m$ is best.

(Note $\|\cdot\|_{2\alpha,\gamma,m,m} = \|\cdot\|_{2\alpha,\gamma,m}$)

④ Controlled spatial Lip_x^2 func.:

Def: i) $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is γ-Lip, $\gamma \in (0,1]$ if $\exists C$

$|f(x_1) - f(x_2)| \le C|x_1 - x_2|^{\gamma}$, $\forall x_1, x_2 \in \mathbb{R}^{d_1}$.

Set $[f]_\gamma := \min\{c > 0 \mid \Rightarrow\}$. $|f|_\gamma = |f|_\infty + [f]_\gamma$.

Inductively $[f]_{k+\gamma} := |Df|_{k-1+\gamma}$. $|f|_{k+\gamma} := [f]_{k+\gamma} + |f|_\infty$.

ii) Consider norm $|x,t|_\alpha := |x| \vee |t|^\alpha$. $\bar{\delta} f(\bar{x}_2, \bar{x}_1) = f(\bar{x}_1) - f(\bar{x}_2)$ for $\bar{x} = (x,t)$. We say $(f, f', \partial f) \in \mathcal{D}_Y^{2\alpha} Lip^2_{x,loc}$ if $\bar{\delta} f \overset{2}{=} f' \delta Y + \partial f \bar{\delta} x$.

$\bar{\delta} f' \overset{!}{=} 0$. $\bar{\delta} \partial f \overset{!}{=} 0$.

And $\mathcal{D}_Y^{2\alpha} Lip_x^2 = \{ f \in \mathcal{D}_Y^{2\alpha} Lip_{x.loc}^2 \mid |f|_\alpha \vee |f|_\infty \vee |\partial f|_\infty < \infty \}$.

<u>Rmk</u>: i) Note that fix t of f. Then: f is x-diff. and $\partial f = Df$. So: for convention, set $(f, f') \in \mathcal{D}_Y^{2\alpha} Lip_x^2$.

ii) Fix x. we see $(f, f')(x, \cdot) \in \mathcal{D}_Y^{2\alpha}$ and fix t. $f(\cdot, t) \in Lip_x^2$ (But $\mathcal{D}_Y^{2\alpha} Lip_x^2$ can't be understood as crp. $\mathcal{D}_Y^{2\alpha}(\mathbb{R}^{\geq 0}; V)$ with $V = Lip_x^2$. i.e.

$$\sup_{t_0, t_1} \frac{\| f(\cdot, t_1) - f(\cdot, t_0) - f'(\cdot, t_0) \delta Y_{t_0 t_1} \|_{Lip_x^2}}{|t_1 - t_0|^{2\alpha}} = \infty$$

may hold since $f(\cdot, t) \in Lip_x^2$ only work for pointwise t).

Lemma 1. *Set* $[(f, f')]_2 := c_{1.1} + c_{1.2} + c'_{1.1} + c'_{1.2}$ *where*

$$c_{1.1} = \sup_t |Df_t|_\infty, c_{1.2} = \sup_t [Df_t]_1, c'_{1.1} = \sup_t |f'_t|_\infty, c'_{1.2} = \sup_t [f'_t]_1$$

and also $[\![f, f']\!]_{Y;2\alpha} := c_{2.1} + c_{2.2} + c'_{2.2} + c_{2.3}$ *where*

$$|f_t - f_s|_\infty \leqslant c_{2.1} |t-s|^\alpha, |Df_t - Df_s|_\infty \leqslant c_{2.2} |t-s|^\alpha, |f'_t - f'_s|_\infty \leqslant c'_{2.2} |t-s|^\alpha, |R^f_{s,t}|_\infty \leqslant c_{2.3} |t-s|^{2\alpha}.$$

Then: $(f, f', Df) \in \mathcal{D}_Y^{2\alpha} Lip_x^2$ with estimate:

$$|Df|_\infty = c_{1.1}, \quad |f'|_\infty = c'_{1.1},$$

and with $(\star) = (|x_1 - x_0| \vee |t_1 - t_0|^\alpha), C_2 = c_{1.2} + c_{2.2}, C_1 = c_{1.2} + c_{2.1}, C'_1 = c'_{1.2} + c'_{2.1}$,

$$|\bar\delta f(\bar x_0, \bar x_1) - f'(\bar x_0)\delta Y_{t_0 t_1} - \partial f(\bar x_0)(x_1 - x_0) \leqslant C_2(\star)^2, |\bar\delta f'(\bar x_0, \bar x_1)| \leqslant C'_1(\star), |\bar\delta \partial f(\bar x_0, \bar x_1)| \leqslant C_1(\star),$$

Rmk: It gives criteria for $(f, f', Df) \in D_Y^{2\gamma} Lip_x^2$.

Pf: Note that $g(x_1, t_1) - g(x_0, t_0) = (g(x_1, t_1) - g(x_0, t_1)) + (g(x_0, t_1) - g(x_0, t_0))$.

Let $g = Df$ and f' then use assumpts.

prop. (composition of crp and (f))

Let $X \in D_Y^{2\gamma}$, $(f, f') \in D_Y^{2\gamma} Lip_x^2$. Then: $(Z, Z') = (f(X), Df(X)X' + f'(X)) \in D_Y^{2\gamma}$.

Pf: $\delta Z(t_0, t_1) = \bar\delta f((X_{t_0}, t_0), (X_{t_1}, t_1))$

$\overset{2}{=} f'(X_{t_0}, t_0)\delta Y_{t_0 t_1} + Df(X_{t_0}, t_0)\delta X_{t_0 t_1}$

$\overset{2}{=} (f' + Df \cdot X')(X_{t_0}, t_0)\delta Y_{t_0 t_1}$

Similarly we have $\delta Z' \overset{1}{=} 0$ by using definition of $D_Y^{2\gamma} Lip_x^2$.

Rmk: For $(f, f') \in D_Y^{2\gamma} Lip_x^2$:

$\delta Z = \delta \int (f(x), Df(x) f(x) + f'(x))\, dY$

$\Rightarrow (f, f')$ is control. vector field as in ODE $dX = f(X)\, dY$.