Rough SDEs

(1) Mixed Moment space:

prop. For $1 \le m \le n \le \infty$

i) L_{mn} is Banach space.

ii) $L_n \hookrightarrow L_{mn} \hookrightarrow L_m$.

iii) (l.s.c.) For $(\xi^k) \subset L_m$ has a.s.-limit ξ

$\Rightarrow \|\xi\|_{mn} \le \varliminf_k \|\xi^k\|_{mn} \le \infty$

iv) $\xi^k \to \xi$ in $L_m \Rightarrow \|\xi\|_{mn} \le \varliminf_k \|\xi^k\|_{mn} \le \infty$

Pf: iii), iv) is from Fatou type inequa.

Cor. $A^k_{\cdot\cdot} \to A_{\cdot\cdot}$ in $C_2^\kappa L_m$. Then we have

$\|A\|_{\kappa,m,n} \le \varliminf_k \|A^k\|_{\kappa,m,n} \le \infty$.

Pf: Apply iv) on $\xi^k_{st} = A^k_{st}$

(2) Well-posedness:

Def: Conti. adapted process X on $[0,T]$ is $L_{m\infty}$-

Sol. w.r.t $\mathbf{Y} = (Y, \mathbb{Y}) \in \mathscr{C}^\alpha$, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$

$dX_t = (f, f')(X_t) d\mathbf{Y}_t + \sigma_t(\omega, X_t) dB_t(\omega) + b_t(\omega, X_t) dt$

in sense of SRI + Itô + Lebesgue where

i) $(f(x), Df(x)f(x) + f'(x)) \in \mathcal{D}_Y^{2\gamma} L_{m,\infty}$

ii) $\sigma\sigma^T, b \in L^1_{loc}([0,T])$ a.s. progressive.

Rmk: Recall Davie expansion of RDE:

$dX_t = V(X_t)\, dY_t$. $V = (V_1 \cdots V_d)$ is

$$X_t - X_s = V_i(X_s) Y^i_{st} + V_j \circ V_i(X_s) \mathbb{Y}^{i,j}_{s,t} + R^x_{s,t}$$

where $|R^x_{s,t}| \lesssim |t-s|^{3\gamma}$.

Due to $3\gamma > 1$ and Sto-sewing, Davie expansion can characterize Sol. of RSDE in Def by:

$$X_t - X_s = f(X_s) Y_{st} + (Df(X_s)f(X_s) + f'(X_s)) \cdot \mathbb{Y}_{st} + \sigma(X_s) B_{s,t} + b(X_s)(t-s)$$

Def: $(f, f') \in \mathcal{D}_Y^{2\gamma} Lip^3_x$ if $(f, f') \in \mathcal{D}_Y^{2\gamma} Lip^2_x$ and $(Df, Df') \in \mathcal{D}_Y^{2\gamma} Lip^2_x$

Rmk: If f is autonomous, i.e. $f = f(x)$, then it's reduced to Lip^3_x.

Assumpt: i) $b, \sigma \in Lip^1_x$ uniform in (t, w) and uniformly bdd in (t, x, w)

ii) $(f, f') \in \mathcal{D}_Y^{2\gamma} Lip^3_x$.

Thm. Under assumpt i) – ii). For any initial datum $\xi \in \mathcal{L}^0$. Then: there exists unique RSDE sol. in $D_Y^{2\alpha} L_{2,\infty}$. Besides, the sol. also belongs to $D_Y^{2\alpha} L_{m,\infty}$. $\forall m < \infty$.

Rmk: i) For sol. in $D_Y^{2\alpha} L_{m,n}$:

$m \uparrow \Rightarrow$ better result for existence.

$m \downarrow \Rightarrow$ better result for unique.

ii) We'll do the Picard iterate in $D_Y^{2\alpha} L_{2,\infty}$ for $\frac{1}{3} < \beta < \alpha$ as we did in RDE or ODE case (sol. of ODE $\in C^1$. But iterate happens in C^0)

Pf: With minimal loss of generality:

Set $g = (f_x, f_x') \in Lip_x^3$ and $b \equiv 0$ (similar estimate as $\square dB$ part)

Denote $dm_t = \sigma_t(w, X_t) dB_t$

$\Rightarrow dX_t = g(X_t) d\mathbb{Y}_t + dm_t$.

Denote $\mathcal{X} = (X, X')$.

$\mathcal{X}^0 = (\xi + g(\xi)\delta Y_{0,\cdot}, g(\xi))$.

$[[\mathcal{X}]]_{\gamma,2p,m,\infty} := \|\delta X\|_{p,m,\infty} + \|\delta X'\|_{\beta,m,\infty} + \|\mathbb{E}.(R^X)\|_{2p,\infty}$

$\|\mathcal{X}\|_{\gamma,2p,m,\infty} = [[\mathcal{X}]]_{\gamma,2p,m,\infty} + \sup_t \|X'_t\|_\infty$.

$\Phi(\mathcal{X}) = (\xi + \int_0^\cdot g(X_s)\,d\mathbf{Y}_s + m_\cdot,\ g(X_\cdot))$

Define $B_T(C) = \{\mathcal{X} \in \mathbf{D}_Y^{2\beta} L_{m,\infty}$ has starting cond. $X_0 = \xi,\ X'_0 = g(\xi)$ and $\|\mathcal{X}\|_{\gamma,2p,m,\infty} \leq C\}$.

pinned rough path ball. $\mathcal{X}^0 \in B_T(C) \neq \emptyset$.

Set $\delta = \gamma - \beta > 0$. As before, we want T small enough to get contraction factor T^δ.

i) Invariance of $B_T(C)$ under Φ:

Set $(Z, Z') = (g(X), (Dg)g(X)) \overset{\text{prop.}}{\in} \mathbf{D}_Y^{2\gamma} L_{m,\infty}$.

And $[[\int Z\,d\mathbf{Y}, Z]]_{\gamma,2p,m,\infty}$

$\lesssim C(\|g\|_\infty + \|Z, Z'\|_{\gamma,2\gamma,m,\infty})\, T^\delta$

$\overset{g\text{ bdd}}{\lesssim} C(\|g\|_\infty + C_g \|X, X'\|_{\gamma,2\gamma,m,\infty})\, T^\delta$.

Note $\mathbb{E}.(R^m) = 0$ since $m' = 0$, $m \in \mathbf{D}^{2+\varepsilon}$.

By BDG: $\|\delta m\|_{\beta,m,n} \leq C_m$.

So we can choose C large and T small enough to get $\Phi(\mathcal{X}) \in B_T(C)$ for $\forall \mathcal{X} \in B_T(C)$.

2) Contraction of Φ:

With Fatou's type Lem. We can prove that $(\mathcal{B}_T(c), [[\cdot,\cdot]]_{Y,2\beta,m})$ is complete.

$$S_0 = [[\Phi(X,X') - \Phi(\tilde{X},\tilde{X}')]]_{Y,2\beta,m}$$

$$\leq \|\delta(m-\tilde{m})\|_{\alpha,m} \cdot T^{\alpha-\beta}$$

$$+ [[(\int Z\,d\mathbf{Y}, Z) - (\int \tilde{Z}\,d\mathbf{Y}, \tilde{Z})]]_{Y,2\beta,m}$$

$$\lesssim C_m T^{\alpha-\beta} + \|Z, Z'; \tilde{Z}, \tilde{Z}'\|_{Y,2\beta,m} T^{\gamma-\beta}.$$

$$\leq T^{\alpha-\beta}(C_m + \|X, X'; \tilde{X}, \tilde{X}'\|_{Y,2\beta,m})$$

$\Rightarrow$ Choose $T>0$ small enough. Then Φ is contraction on $\mathcal{B}_T(c)$.

3) Uniqueness: $\forall m$. $D_Y^{2\gamma}L_m$ - sol. is $D_Y^{2\gamma}L_{2m}$.

Rmk: i) Actually. We'd like to introduce $\beta' < \beta$ and consider $(X, X') \in D_Y^{\beta,\beta'}$ to derive the contract factor

ii) If $(Y, \mathbb{Y}) \in \mathscr{C}^\gamma$ is canon. lift of C^1 -path Y. i.e. $\mathbb{Y} = \int Y\,dY$. We see: $g(X)\,d\mathbf{Y} = g(X)Y'\,dt$ now. So the RSDE sol. is SDE-sol. which can be considered as some approxi. for RSDE.

Thm. For $(Y, \mathbb{Y}) \in \mathscr{C}_g^{0,r}$. Assume Y^ε is C^∞-approxi. for Y. And X^ε satisfies SDE:

$$dX_t^\varepsilon = \sigma_t(w, X_t^\varepsilon)\, dB_t + (g(X_t^\varepsilon)\, \dot{Y}^\varepsilon + b(w, X_t^\varepsilon))\, dt$$

Then: $X^\varepsilon \xrightarrow{ucp} X$. sol. of RSDE w.r.t $\mathbf{Y}$.

Thm. (Stability)

If $X_t, \bar{X}_t$ are sol. of RSDE driven by data $X_0 = \xi, b, \sigma, g = (f, f')$ and $\bar{X}_0 = \bar{\xi}$, $\bar{b}, \bar{\sigma}, \bar{g} = (\bar{f}, \bar{f}')$, $\bar{\mathbf{Y}}$. Then:

$$\|\sup_{t \le T} |\delta X_{st} - \delta \bar{X}_{st}|\|_m + \|X, f(X); \bar{X}, \bar{f}(\bar{X})\|_{Y, \bar{Y}, 2r, m}$$

$$\lesssim \| |\xi - \bar{\xi}| \wedge 1 \|_m + \sup_{t \le T} \|\sup_x |b - \bar{b}|\|_m + \|g - \bar{g}\|_{Lip^2}$$

$$+ \sup_{t \le T} \|\sup_x |\sigma - \bar{\sigma}|\|_m + \|\delta Y - \delta \bar{Y}\|_r + \|\mathbb{Y} - \bar{\mathbb{Y}}\|_{2r}$$

(3) <u>Itô's formula:</u>

Def: Second controlled path $(X, X', X''; \dot{X}) \in D_Y^{3r}$ if $\delta X \overset{3r}{=} X' \delta Y + X'' \mathbb{Y} + \dot{X} \delta t$ and $(X', X'') \in D_Y^{2r}$

Rmk: Sol. of RDE $dX = (f, f')(X)\, d\mathbf{Y} + b(X)\, dt$ is naturally 2nd-controlled path

(think $X' = f(X)$, $X'' = f'(X) X' + Df(X) \dot{X}$, $X'\, dV = f b(X)\, dt$)

Lem. If $3\gamma > 1$. Then: $(X, X', X''; \dot{X}) \in \mathcal{D}_Y^{3\gamma} \Leftrightarrow (X', X'') \in \mathcal{D}_Y^{2\gamma}$ and $dX = (X', X'') dY + \dot{X} dt$

Lem. If $3\gamma > 1$. $(X, X', X''; \dot{X}) \in \mathcal{D}_Y^{3\gamma}$, $g \in Lip^3$. $[Y] \in C^1$ with deri. $[\dot{Y}]$. Then: $(Z = g(X), Z' = Dg(X)X', Z'' = D^2 g(X)(X', X') + Dg(X)X''; \dot{Z} = \frac{1}{2} D^2 g(X)(X', X')[\dot{Y}] + Dg(X)\dot{X}) \in \mathcal{D}_Y^{3\gamma}$ as well.

Prop. (Rough Itô formula)

Under the same conditions as above $\Rightarrow$

$$dg(X) = (Z', Z'') dY + \frac{1}{2} D^2 g(X)(X', X') d[Y] + Dg(X)\dot{X} dt$$

Add Stochastic: Consider $(X, X', X''; M, V)$ where $dM = \sigma_t(\omega) dB_t$, $dV = b_t(\omega) dt$. $(X', X'') \in \mathcal{D}_Y^{2\gamma} L_{m,n}$, s.t. $dX_t = (X', X'')_t dY_t + dM_t + dV_t$.

Thm. (Rough Stochastic Itô)

If b, σ are bdd, progressive, m large enough, $\sup_{t \le T} \| X_t' \|_m < \infty$. $g \in Lip^3$. Then:

$$dg(X) = \mathcal{L}_{t,\omega} g(X) dt + Dg(X_t) dM_t + (Dg(X)X', D^2 g(X)(X', X') + Dg(X)X'') dY_t + \frac{1}{2} D^2 g(X)(X', X') d[Y]_t$$

where $\mathcal{L}_{t,w} = b_t \cdot D + \frac{1}{2}\sigma_t\sigma_t^T \cdot D^2$ generator.

<u>Pf</u>: By cond. $\Rightarrow$ we have (X, X') is jointly controlled w.r.t. $J = (Y, m)$ with Gubinelli derivative $(Y', 1)$. (Ignore dV_t)

Besides. $[J] = \begin{pmatrix} [Y] & 0 \\ 0 & [m] \end{pmatrix}$ by $[Y, m] = 0$ from $Y \in C^{\alpha}$, $m \in V^{2+\varepsilon}$

Insert them into rough Itô's formula.

Next. We consider to get a formula for more general g:

Still consider on the norm $|X, t| = |t|^{\alpha} \vee |X|$ and recall we've defined $D_Y^{2\alpha} Lip^2$ before.

<u>Def</u>: $g = (F, F', \partial F, F'', \partial F', \partial^2 F, \dot{F}) \in D_Y^{3\alpha} Lip^3$ if

i) $\bar{\delta} F \overset{3}{=} F' \delta Y + \partial F \delta X + F'' \mathbb{Y} + \frac{1}{2}\partial^2 F (\delta X, \delta X)$
$+ \partial F' (\delta X, \delta Y) + \dot{F} \delta t$

ii) $\bar{\delta} F' \overset{2}{=} F'' \delta Y + \partial F' \delta X$. $\bar{\delta} \partial F \overset{2}{=} \partial^2 F \delta X$

iii) $\bar{\delta} \partial F' \overset{1}{=} 0$, $\bar{\delta} \partial^2 F \overset{1}{=} 0$. $\bar{\delta} F'' \overset{1}{=} 0$.

Consider RIP $dX = (X', X'') d\mathbf{Y} + dm + dV$ and

$Z = F(X)$. $Z' = DF(X) X' + F'(X)$. $Z'' = D^2 F(X)(X', X')$
$+ DF(X) X'' + DF'(X) X' + F''(X)$ where $F \in D_Y^{3\alpha} Lip_x^3$

Thm. (Rough Stoch. Itô Wentzell)

For $Z = F(X)$, we have:

$$dZ = (Z', Z'')d\mathbf{Y} + dM^Z + dV^Z, \text{ where}$$

$$dM^Z = DF(X)dM.$$

$$dV^Z = DF(X)dV + \tfrac{1}{2}D^2F(X)d[M] + (DF'(X)X' + \tfrac{1}{2}D^2F(X)(X', X'))d[\mathbf{Y}] + \dot{F}(X)dt$$

Pf: By $dX = \square$. $\Rightarrow \delta X \overset{2}{\approx} (X', 1)\begin{pmatrix}\delta Y \\ \delta m\end{pmatrix}$. i.e.

X is controlled by $(\mathbf{Y}, m)$.

Denote $\bar{\mathbf{Y}} = (Y, m)$. lift $\bar{\mathbb{Y}} = \begin{pmatrix} \mathbb{Y} & \int \delta Y dm \\ \int \delta m dY & \int \delta m dm \end{pmatrix}$

And $\delta X = X'\delta Y + X''\mathbb{Y} + \delta M + \delta V$.

$$\delta Z \overset{3}{=} F'(X)\delta Y + DF(X)\delta X + F''(X)\mathbb{Y} + \dot{F}(X)dt + \tfrac{1}{2}D^2F(X)(\delta X, \delta X) + \partial F'(\delta X, \delta Y)$$

by def of $F \in D_Y^{2\alpha} Lip_x^3$.

Then insert δX into δZ to get it.

(4) Rough Feynman-Kac:

Consider sol. $X^{x,s,\mathbf{Y}} =: X$ for RSDE:

$$dX_t = (f_t, f_t')(X)d\mathbf{Y}_t + \sigma_t(X_t)dB_t + b_t(X_t)dt.$$

$X_s = x$. $t \geq s$.

WLOG, assume $[\mathbf{Y}] = 0$. i.e. $\mathbf{Y} \in \mathscr{C}_g^\alpha$. otherwise

We can consider $b_t(\cdot) - [Y]_t = \tilde{b}_t$.

Define $\Gamma V := f \cdot DV$, $\Gamma' V := f' \cdot DV$ and L is generator of $\sigma dB_t + b\,dt$.

Consider $u(t,x)$ solves backward RPDE:

$-d_t u_t = \Gamma_t u_t \circ dY_t + L_t u_t\, dt$, $u(T,x) = g(x)$.

$(\Gamma_t u)' = \Gamma_t' u_t - \Gamma_t^2 u_t$ ("$\circ$" means $Y \in \mathscr{C}_g^\alpha$)

Rmk: Let $f_t = 0$, $u \in C_c^2$. Then by Itô's formula, we have $u(t, X_t^{s,x}) \in M_c^{loc}$ as in Itô calculus case.

We say u is sol. to this RPDE if:

$u \in D_Y^{3\alpha} Lip^3$ and $-u' = \Gamma u$, $-u'' = \Gamma' u - \Gamma^2 u$, $-\dot{u} = Lu$.

Rmk: i) We see by def of $D_Y^{3\alpha} Lip^3$, u will satisfies the RPDE

ii) u is necessarily $\in C_x^2$, $\partial u = D_x u$, $\partial^2 u = D_x^2 u$.

iii) Motivation of def of $-u'$, $-\dot{u}$ is obvious, while $-u'' = (\Gamma_t u)' = \Gamma_t' u + \Gamma_t u' = -\Gamma_t' u + \Gamma_t(\Gamma_t u)$ where $\Gamma_t' u \approx \Gamma_{t,t+\Delta t} u = f_{t,t+\Delta t}' Du \approx f_t' Du$.

(f' is Gubinelli deri. w.r.t. Y.)

$\Rightarrow$ Apply RSIW formula on $u(t, X_t^{s,x})$, we can also see $u(t, X_t) \in \mathcal{M}_c^{loc}$.

For uniqueness of the RPDE is from:

$u(s,x) = \mathbb{E}(u(T, X_T^{s,x})) = \mathbb{E}(g(X_T^{s,x}))$

(5) <u>Fokker-Planck equation:</u>

Denote $\mu_t = \text{Law}(X_t)$, X_t is sol. of the RSDE.

Consider $\varphi \in \text{Lip}_x^2$. Apply rough stoch. Itô's formula on $\varphi(X_t)$. We have:

$d\langle \mu_t, \varphi \rangle = \langle \mu_t, \Gamma\varphi \rangle \, d\mathbf{Y}_t + \langle \mu_t, L_t\varphi \rangle dt$

i.e. FPE $d\mu_t = \Gamma^* \mu_t \, d\mathbf{Y}_t + L^* \mu_t \, dt$.

Sol. to this FPE is unique if there exists a regular sol. for RPDE:

$-d_t u_t = \Gamma u_t \, d\mathbf{Y}_t + L u_t \, dt$, $u(T,x) = \varphi(x)$, $\forall \varphi \in C_c^\infty$.

By duality argument: if μ, $\tilde{\mu}$ are both sol.'s

$\Rightarrow \partial_t \langle \mu_t, u_t \rangle = \langle \partial_t \mu_t, u_t \rangle + \langle \mu_t, \partial_t u_t \rangle$

$= \langle \mu_t, \partial_t u_t - \Gamma u_t \, d\mathbf{Y}_t - L u_t \, dt \rangle = 0$.

So: $\langle \mu_t, u_t \rangle \equiv C$. $\Rightarrow \langle \mu_t - \tilde{\mu}_t, u_t \rangle = 0$.

i.e. $\langle \mu_T - \tilde{\mu}_T, \varphi \rangle = 0$, (let $t = T$). $\forall \varphi \in C_c^\infty$.

$\Rightarrow \mu_T = \tilde{\mu}_T$, $\forall T > 0$. i.e. $\mu = \tilde{\mu}$.