Applications of RSDEs

(1) Rough Stoch. Filtering:

Recall in our system:

$dY_t = h(X_t, Y_t)dt + dB_t^{\perp}$. (Observation)

$dX_t = b(X_t, Y_t)dt + \sigma(X_t, Y_t)dB_t(w) + \gamma(X_t, Y_t)\circ dY_t$

Recall (B, Y) are 2-dim BM. under $\mathbb{P}^0$ where

$d\mathbb{P}^0/d\mathbb{P}|_{\mathcal{G}_t} = \exp(\int_0^t h(X_s, Y_s)\circ dY_s - \frac{1}{2}\int_0^t h^2(X_s, Y_s)ds)$

And Kallianpur-Striebel formula hold for

filter $\mathbb{E}^{\mathbb{P}^0}(\varphi(X_t) | \mathcal{F}_t^Y) = \mathbb{E}(\varphi(X_t)\frac{d\mathbb{P}^0}{d\mathbb{P}}|_{\mathcal{G}_t} | \mathcal{F}_t^Y)$

$=: \langle \mu_t, \varphi \rangle$. Let $\langle \pi, \varphi \rangle \overset{\text{nor.}}{=} \frac{\langle \mu_t, \varphi \rangle}{\langle \mu_t, 1 \rangle}$.

Since Y is just observation and we can freeze it to get $\bar{X} = X^Y|_{Y = \mathbb{Y}^{\text{strato}}(w)}$ with RSDE

$X_t^Y - X_0 - \int_0^t b(X_s^Y, Y_s)ds - \int_0^t \sigma(X_s^Y, Y_s)dB_s(w) =$

$\int_0^t (f_s(X_s), (f_s(Df_s) + f_s')(X_s))d\mathbb{Y}_s$

where $f_t(x) = \gamma(x, Y_t)$. $f_t'(x) = D_y\gamma(x, Y_t)$.

Rmk: Such procedure is "randomization". Note $(Y(w))_{w\in\Omega}$ is infinite-dim. So we need to consider its measurability which can be achieved by measurable selection.

Thm. (Robust of filter)

The formula still holds for regular φ:

$$\begin{aligned}\langle\mu_t,\varphi\rangle(\omega) &= \mathbb{E}\Big(\varphi(X_t^Y)\exp\Big(\int_0^t h(X_s^Y,Y_s)dY_s+\int_0^t \underbrace{c(X_s^Y,Y_s)}_{:=-\frac{1}{2}h^2(\cdots)}ds\Big)\Big)\Big|_{Y=Y^{\mathrm{Strato}}(\omega)}\\ &=: \langle\mu_t^Y,\varphi\rangle|_{Y=Y^{\mathrm{Strato}}(\omega)}\end{aligned}$$

And $Y\mapsto\langle\mu_t^Y,\varphi\rangle$ is locally Lip.

Pf: By John-Nirenberg esti.: $\exp(\cdots)\in L'$.

Robust is from def of SRI.

Thm. (Zakai)

$\nu_t=\mu_t^Y$ is unique sol. to: $\nu_0=\mathrm{law}(X_0)$.

$$\begin{aligned}\langle\nu_t,\varphi\rangle-\langle\nu_0,\varphi\rangle-\int_0^t\langle\nu_s,L_s^Y\varphi\rangle\,ds &= \int_0^t\langle\nu_s,(\Gamma_s^Y,\Gamma_s'^Y)\varphi\rangle\circ dY_s\\ &\equiv \int_0^t(\langle\nu_s,\Gamma_s^Y\varphi\rangle,\langle\nu_t,(\Gamma_t^Y\Gamma_t^Y+\Gamma_t'^Y)\varphi\rangle)\circ dY_s.\end{aligned}$$

where
$$\begin{aligned}L_t^Y\varphi(x) &= \tfrac{1}{2}D^2\varphi(x):(\sigma\sigma^\dagger)(x,Y_t)+D\varphi(x)b(x,Y_t)+c(x,Y_t)\\ \Gamma_t^Y\varphi(x) &= D\varphi(x)\gamma(x,Y_t)+\varphi(x)h(x,Y_t)\\ \Gamma_t'^Y\varphi(x) &= D\varphi(x)D_y\gamma(x,Y_t)D\varphi+\varphi(x)D_yh(x,Y_t)\end{aligned}$$

for φ regular enough

Pf: Apply rough stoch. Itô formula on $\varphi(X_t^Y)\exp(\cdots)$. And uniqueness is from duality argument.

Rmk: Compared to BSPDE duality method

there's no sobolev embedding. So it avoids requirement on regular coefficients when d is large.

(2) Rough Stoch. Control:

For $X = X^{\eta(\omega)}$ solves controlled RSDE: $\mathbf{Y} \in \mathcal{C}^\alpha$.

$$dX_t = f(X_t)d\mathbf{Y}_t + \sigma(X_t, \eta_t(\omega))dB_t(\omega) + b(X_t, \eta_t(\omega))dt$$

Denote: $V(s, y; \mathbf{Y}) = \inf_{\eta(\cdot)} \mathbb{E}^{s,y}(g(X_T) + \int_s^T \ell(t, X_t; \eta_t(\omega))dt$

value func.

Thm. (Regularity of V)

For (f, σ, b, g, ℓ), $(\bar f, \bar\sigma, \bar b, \bar g, \bar\ell)$ bdd. Lip.

Then: $|\bar{\mathcal{V}}(\bar t, \bar y; \bar{\mathbf{Y}}) - \mathcal{V}(t, y; \mathbf{Y})| \lesssim |(\bar g - g)_+|_\infty + |T - t \vee \bar t|^\alpha \{(1) + (2) + (3)\}$

where

$$\begin{aligned}
(1) &= |(b, \sigma, \ell) - (\bar b, \bar\sigma, \bar\ell)|_\infty + |f - \bar f|_{C_b^{\alpha+}} \\
(2) &= \|Y - \bar Y\|_\alpha + \|\mathbb{Y} - \bar{\mathbb{Y}}\|_{2\alpha} \\
(3) &= |t - t|^\alpha + |y - \bar y|
\end{aligned}$$

Thm. (Rough DPP)

$$\mathcal{V}(s, x; \mathbf{Y}) = \inf_{\eta(.)} \mathbb{E}^{s,x}\left(\mathcal{V}(s+h, X_{s+h}; \mathbf{Y}) + \int_s^{s+h} \ell(t, X_t; \eta_t)dt\right)$$

Pf: WLOG. let $\mathbf{Y} \in \mathcal{C}_g^{\alpha,0}$. So $\exists\, Y^\varepsilon$ smooth $\xrightarrow{\mathcal{C}^\alpha} Y$. Then use conti. of $\mathbf{Y} \mapsto V$ and result is from classical DPP for V^{Y^ε}.

Cor. If $(Y^\varepsilon, \int \delta Y^\varepsilon d Y^\varepsilon) \xrightarrow{\mathscr{C}^\alpha} \mathbf{Y}$. Then: s.l. to

$$-\partial_s \mathcal{V}^\varepsilon = H(s,x,D\mathcal{V}^\varepsilon(\star), D^2\mathcal{V}^\varepsilon) + (f(x)\cdot D\mathcal{V}^\varepsilon)\dot{Y}_s^\varepsilon, \qquad \mathcal{V}^\varepsilon(T,\cdot) \equiv g$$

converges uniformly to $\mathcal{V}(\cdot,\cdot;\mathbf{Y})$

Rmk: It gives a limit meaning to rough HJB equation:

$$d_s\mathcal{V} = H(s,x,D\mathcal{V},D^2\mathcal{V})dt + (f(x)\cdot D\mathcal{V})d\mathbf{Y}_t.$$

Let $\ell = 0$. We can compare with classical stoch. control problem:

$$\mathcal{V}(s,x;\mathbf{Y}) \equiv \inf_{\eta(.)} \mathbf{E}^{s,y}(g(X_T^{\mathbf{Y},\eta})) \quad \rightsquigarrow \quad V(t,x;\omega) \equiv \operatorname{essinf}_{\nu(.)} \mathbf{E}^{t,x}(g(X_T^\nu)|\mathcal{F}_T^W)$$

Thm. $\mathcal{V}(s,x;\mathbf{Y})|_{\mathbf{Y}=W(\omega)} = V(t,x;W)$ a.s. for admissible choice $\nu(\cdot)$, where W is BM.

(3) Mckean-Vlasov RSDEs:

Consider n particles system with common noise $W_t \perp B_t^i$, $\forall i$, $\mu_t^n := n^{-1}\sum_i^n \delta X_t^{n,k}(\omega) \in \mathcal{P}(C([0,T])$

$$dX_t^{n,i} = f(X_t^{n,i}, \mu_t^n) dW_t + \sigma(X_t^{n,i}, \mu_t^n) dB_t^i + b(X_t^{n,i}, \mu_t^n) dt$$

Denote empirical mean $m_t^n(\omega) = \frac{1}{n}\sum_{k=1}^n X_t^{n,k}(\omega)$

Next, we replace W by rough common noise:

$$dX_t^{n,i} = (f_t^n, f_t^{n\prime})(X_t^{n,i}) d\mathbf{Y}_t + \sigma(X_t^{n,i}, \mu_t^n) dB_t^i + b(\cdots)dt.$$

with $f_t^n(x) := f(x, \mu_t^n)$, $\mathbf{Y}_t \in \mathscr{C}^\alpha$ deterministic.

Let $n \to \infty$. We expect it converges to MKV-RSDE: $dX_t = \langle f_t, f_t' \rangle (X_t) \, dI_t + \sigma(X_t, \mu_t) \, dB_t + b(X_t, \mu_t) dt$. where $X_t \sim \mu_t$.

e.g. Consider $dX_t^{n,i} = dY_t + dB_t^i + a(m_t^n - X_t^{n,i}) dt$

$X_0^{n,i} = \xi^i \in L^1$. i.i.d. (So: $m_0^n \xrightarrow{LLN} E(\xi^1)$). And Y_t is deterministic func.

Note $m_t^n = m_0^n + Y_t + \frac{1}{n}\sum_1^n B_t^i + a\int_0^t m_s^n - m_s^n \, ds$

$\xrightarrow{a.s.} \mathbb{E}(\xi) + Y_t$ by LLN.

By basic limit Thm for MKV:

We have: $X^{n,i} \Rightarrow X^i$. where X^i satisfies

$X_t^i = \xi^i + Y_t + B_t^i + a\int_0^t (\mathbb{E}(\xi) + Y_s - X_s^i) ds$

$\Rightarrow A_t := \mathbb{E}(X_t^i - \xi^i) - Y_t = a\int_0^t A_s ds$

So: $A_t = 0$. i.e. $\mathbb{E}(X_t^i) = \mathbb{E}(\xi) + Y_t$. Then:

$X_t^i = \xi^i + Y_t + B_t^i + a\int_0^t (\mathbb{E}(X_s^i) - X_s^i) ds$

Note if Y is replaced by W_t. SBM. repeat the argument above. then we can still get the same result in $\mathbb{E}(\cdot \mid \mathcal{F}_t^W)$:

i.e. $X_t = \xi + W_t(\omega) + B_t(\omega) + a\int_0^t (\mathbb{E}(X_s \mid \mathcal{F}_s^W) - X_s) ds$

Rmk: Let $Y = W$ is kind of randomization

So we can't directly get the result.

Sometimes, there's problem:

e.g. If $Y \in C[0,T]$ deterministic and W is SBM. Then:

$\langle Y, W \rangle = \lim \sum \delta Y \delta W = 0.$

Since $|\sum \delta Y \delta W| \le \|Y\|_{V^1} \cdot \sup_i |\Delta W_i| \xrightarrow{a.s.} 0$

But $\langle Y, W \rangle|_{Y=W} = t \ne 0.$

Recall in the RSDE case, we require measurable selection.

① For $dX_t^\nu = (f_t, f_t')(X_t^\nu) d\mathbf{Y} + \sigma(X_t^\nu, \nu_t) dB_t + b(\cdots) dt$

Next. We assume $\mathbf{Y} \in \mathscr{C}_g^\alpha$. apply rough stoch Itô formula, we have: $\mu_t = \mathrm{Law}(X_t^\nu)$ satisfy

$$d\langle \mu_t, \varphi\rangle = \langle \mu_t, (\Gamma_t, \Gamma_t')\varphi\rangle \circ d\mathbf{Y}_t + \langle \mu_t, (L_t^\nu \varphi)\rangle dt \quad \Leftrightarrow \quad d\mu_t = (\Gamma_t, \Gamma_t')^* \mu_t \circ d\mathbf{Y}_t + (L_t^\nu)^* \mu_t dt;$$

$$L_t^\nu \varphi(x) = D\varphi(x) b(x, \nu_t) dt + \tfrac{1}{2} D^2\varphi(x) : (\sigma\sigma^\dagger)(x, \nu_t), \Gamma_t\varphi(x) := D\varphi(x) f_t(x), \Gamma_t'\varphi(x) := D\varphi(x) f_t'(x).$$

where $\langle \mu_t, \varphi \rangle = \mathbb{E}(\varphi(X_t^\nu))$

From limit view: if $\varphi \in \mathrm{Lip}_x^3$, we have:

$$\begin{aligned}
d\varphi(X_t^{n,i}) &= D\varphi(X_t^{n,i}) f_t(X_t^{n,i}) d\mathbf{Y}_t \\
&+ D\varphi(X_t^{n,i}) \sigma(X_t^{n,i}, \mu_t^n) dB_t^i \\
&+ \{D\varphi(X_t^{n,i}) b(X_t^{n,i}, \mu_t^n) dt + \tfrac{1}{2} D^2\varphi(X_t^{n,i}) : (\sigma\sigma^\dagger)(X_t^{n,i}, \mu_t^n)\} dt \\
&\equiv (\Gamma_t\varphi, \Gamma_t'\varphi)(X_t^{n,i}) d\mathbf{Y}_t + dM_t^{n,i} + (L^{\mu_t^n}\varphi)(X_t^{n,i}) dt
\end{aligned}$$

where $M_t^{n,i} = \int_0^t \square \, dB_s^i$. Let $M^n = \frac{1}{n}\sum_i^n M^{n,i}$.

Average $i = 1 \cdots n$ as before. We have:

$d\langle \mu_t^n, \varphi\rangle = \langle \mu_t^n, (\Gamma_t, \Gamma_t')\varphi\rangle \circ dY_t + dM_t^n + \langle \mu_t^n, L^{\mu^n}\varphi\rangle dt$

For $D\varphi, \sigma$ bdd $\Rightarrow \mu^n$ is mart. st. $\langle \mu^n\rangle \sim \frac{1}{n}$

Let $n \to \infty$. We have:

$d\langle \mu_t, \varphi\rangle = \langle \mu_t, (\Gamma_t, \Gamma_t')\varphi\rangle \circ dY_t + \langle \mu_t, L^{\mu_t}\varphi\rangle dt.$

Uniqueness of sol. is from duality argument

Rmk: Similarly for $V(t, x) \in D_Y^{3\gamma} Lip_x^3$. st.

$-V' = \Gamma_t V_t, \quad -V'' = \Gamma_t' V_t - \Gamma_t^2 V_t$

$\Rightarrow d_t V_t = (\Gamma_t, \Gamma_t')V_t \circ dY_t + L_t V_t dt.$

And $V(t, X_t^Y)$ is mart. $V(t, x) = \mathbb{E}^{t,x}(V(T, X_T))$

② If we consider the rough term depends on measure as well, i.e.

$dX_t^v = (g, g')(X_t^v, V_t) \circ dY_t + \sigma(X_t^v, V_t)dB_t + b(\cdots)dt$

It causes a harder problem.

WLOG. we consider $dX_t = (f_t, f_t')(X_t^v, V_t) \circ dY_t$

i) For n particles system: $\mu_t^n = \frac{1}{n}\sum_1^n \delta x_t^j$

$dX_t^j = (f_t, f_t')(X_t^j, \mu_t^n) \circ dY_t$. $X_0^j = \xi^j(w) \sim \xi \in L^0$.

We need to understand $f_t'(x, \mu)$, it can be done by Lions lifts:

Denote $\vec{X}_t/ = (X_t^1(w) \dots X_t^n(w))_{w\in\Omega}$. We have.

$f_t(x;\mu_t^n) = \hat{g}(x, \vec{X}_t/)$. Assume $\hat{g} \in Lip^3$ for the well-posedness of the system, $X^i \in L^2$.

$$\Rightarrow f_t'(x, \mu_t^n) = \sum_1^n D_{x^j}\hat{g}(x, \vec{X}_t/)(X_t^j)'$$

$$= \langle \vec{D}\hat{g}(x, \vec{X}_t/), \hat{g}(X_t^j, \vec{X}_t/) \rangle$$

So: $dX_t^j = (\hat{g}(X_t^j; \vec{X}_t/), ((D_x\hat{g})\hat{g}(X_t^j; \vec{X}_t/) + \langle \vec{D}\hat{g}(X_t^j; \vec{X}_t/), \hat{g}(X_t^j; \vec{X}_t/)\rangle)) \circ dY_t$.

ii) Let $n \to \infty$. go back to our MKV-RSDE similarly as above. We can consider Lions lift $\hat{g}(x, Z) = g(x, \mu)$ defined on $\mathbb{R}^d \times L^2(\Omega)$ for $Z \sim \mu$. (Assume $g(x, \cdot)$ on W^2)

<u>Rmk</u>: Notice that most func $g(\cdot)$ defined on W^2 doesn't have regularity. e.g.

$g(\mu) = \int \sin(x)\mu(dx)$. $\hat{g}(Z) = \mathbb{E}(\sin(Z))$

For $H, \tilde{H} \in L^2$. We have:

$\langle D\hat{g}(Z), H\rangle = \mathbb{E}(\cos(Z)H)$

$\langle D^2\hat{g}(Z), (H, \tilde{H})\rangle = \mathbb{E}(-\sin(Z)H\tilde{H})$

$\| D^2\hat{g}(Z) - D^2\hat{g}(\tilde{Z})\| =$

$\sup\{ | \mathbb{E}[(\sin(z) - \sin(\tilde{z})) H\tilde{H}] | : \|H\|, \|\tilde{H}\| \le 1 \}$

$= \| \sin(z) - \sin(\tilde{z}) \|_\infty$

(By Hölder : LHS $\le \| \sin(z) - \sin(\tilde{z}) \|_\infty$. And

we can let $H = H' = I_{A_\varepsilon} / \mathbb{P}(A_\varepsilon)^{\frac{1}{2}}$. where

$A_\varepsilon = \{ w \mid |f(w)| \ge \|f\|_\infty - \varepsilon \}$. Let $\varepsilon \to 0$)

It doesn't converge to 0 for $n \to \infty$ if

$\mathbb{P}(\tilde{z}^n = \frac{z}{2}) = \frac{1}{n} = 1 - \mathbb{P}(\tilde{z}^n = 0)$ and $z = 0$.

<u>Rmk</u>: We see the problem arises cause

$HH' \in L'$ only. If we assume H,

$H' \in L^r$, $z_n \xrightarrow{L^r} z$, $r > 4$. Then $D^2 \hat{j}(z)$

can be conti.

But we can actually require regularity on

its Lions derivative $\partial_\mu j(\mu)(z)$

<u>Rmk</u>: Note $\langle D\hat{j}(z), H \rangle = \mathbb{E}(\cos(z) H)$ z is

r.v. while $\partial_\mu j(\mu)(z) = 2z$, $z \in \mathbb{R}^d$ in the

last example.

iii) Solving MKV-RSDE : We can just consider

$j(X, \mu) = \hat{j}(X, X_t/)$ and apply the Picard

iteration to solve it in moment space.

Now it's controlled in sense:

$\delta g(X_t; X_t/) \overset{2\gamma}{=} D_x g(\cdots)\delta X_{st} + D_2 g(\cdots)\delta X_{st}/.$

Alternatively, we can also freeze $g(X_t, \mu_t^x)$ and replace μ_t^x by another serp. (z, z').

(like replace $Law(X_t)$ by $\mu \in \mathcal{P}(C([0,T])))$

Then consider $\tilde{g}(X_t; (z, z')/)$ as Gubinelli deri. of $X^{(z,z')}$. We find the fix pt of

$(z, z') \mapsto (X^{(z,z')}, \tilde{g}(X_t; (z, z')/))$.