Local Stoch. Vol.

Recall most asset-price models has dynamic: $dX_s^{t,x} = f(s, X_s^{t,x}) V_s dW_s + g(s, X_s^{t,x}) V_s dB_s$, for $0 \le t \le s \le T$. Where (B, W) is 2-dim BM. V_s is vol. f, g are regular so it admits unique strong sol. Next, set $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^B$.

<u>Rmk</u>: (f, g) is chosen for

a) Correlation b) Leverage effect

c) Calibration to IV surfaces.

<u>e.g.</u> $f(t,x) = \sigma(t,x) \rho$.

$g(t,x) = \sigma(t,x)(1-\rho^2)^{\frac{1}{2}}$.

with leverage func. $\sigma(t,x)$ and correlated para. ρ.

<u>Assumpt</u>: Vol. $(V_s)_{[0,T]}$ is bdd. adapted to $\mathcal{F}_s^W$

<u>e.g.</u> Heston, Bergomi.

To solve the pricing problem, we're going to compute $\mathbb{E}(\phi(X_T^{0,x}))$. Where $\phi \in C[0,T]$ is the payoff func.

Rmk: The direct use of MC method won't make use of the structure of the model.

i) Note V_s is totally specified by $\mathcal{F}^W \Rightarrow (X_s^{t,x} \mid \mathcal{F}_T^W$ will be time inhomo. $\mathcal{F}^B$-Markov model.

ii) So to compute

$\mathbb{E}(\varphi(X_T^{0,x})) = \mathbb{E}(\mathbb{E}(\varphi(X_T^{0,x}) \mid \mathcal{F}_T^W))$

$\Rightarrow$ realize V_s and compute

$\mathbb{E}(\varphi(X_T^{0,x}) \mid \mathcal{F}_T^W)$, then take average.

Rmk: It disentangles Vol. V_s from nonlinear local Vol. func. f and g and second BM B.

Next, we want to give the interpretation of "$(X_s^{t,x} \mid \mathcal{F}_T^W = \square$", i.e. give rigourous meaning to dW_t-part:

Fix realization $Y_t = \int_0^t V_s \, dW_s(\omega)$, we interpret it into RSDE:

$dX_s^{t,x,\mathbf{Y}} = f(s, X_s^{t,x,\mathbf{Y}}) d\mathbf{Y}_s + g(s, X_s^{t,x,\mathbf{Y}}) V_s^{\mathbf{Y}} dB_s$

where $\mathbf{Y} = (Y, \mathbb{Y})$. rough path with bracket

$[\mathbf{Y}]_t = \int_0^t (V_s^{\mathbf{Y}})^2 ds$.

Rmk: For any deterministic RP $\mathbf{Y}$ $\Rightarrow$

$X^{t,x,\mathbf{Y}}$ is time-inhomo $\mathcal{F}^B$-Markovian.

Results: i) $\mathcal{L}(X^{t,x} \mid \mathcal{F}_T^W \vee \mathcal{F}_t^B) = \mathcal{L}(X^{t,x,\mathbf{Y}})\big|_{\mathbf{Y} = (\int v dB, \cdots)}$

$\Rightarrow \mathbb{E}(\varphi(X^{0,x})) = \mathbb{E}(\mathbb{E}(\varphi(X^{0,x,\mathbf{Y}})(\omega)))$

ii) $u^{\mathbf{Y}}(t,x) := \mathbb{E}(\varphi(X_T^{t,x,\mathbf{Y}}))$ satisfy some RPDE if considering vanilla option.

Rmk: Such approach can also cover local rough volatility model. even in hyper-rough case. Since we can view $\int v dB$ as rough integral. (v has little regular)

(1) Premilinary:

① Rough Path:

prop. $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{Y} = (Y, \mathbb{Y}) \in \mathscr{C}_g^{\alpha}([0,T], \mathbb{R}^d)$. $\Rightarrow$

$\exists\, Y^{\varepsilon} \in Lip([0,T], \mathbb{R}^d)$. s.t.

$\mathbf{Y}^{\varepsilon} = (Y^{\varepsilon}, \int_0^{\cdot} Y_{0,\cdot}^{\varepsilon} \otimes dY_{\cdot}^{\varepsilon}) \xrightarrow{w} \mathbf{Y}$ on $[0,T]$. $(*)$

with estimate $\sup_\varepsilon (\|Y^\varepsilon\|_\alpha + \|\mathbb{Y}^\varepsilon\|_{2\alpha}) < \infty$. $(*^b)$

Rmk: i) $\mathbf{Y}^\varepsilon$ is so-called geodesic approxi.

ii) We call $\mathbf{Y}^\varepsilon \to \mathbf{Y}$ weakly in $\mathcal{C}^\alpha$ if $(*^a)$ and $(*^b)$ hold.

Lem. $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{Y} \in \mathcal{C}^\alpha$ can be identified uniquely by pair $(\mathbf{Y}^g, [\mathbf{Y}])$ where $\mathbf{Y}^g = (Y, \mathbb{Y}^g) \in \mathcal{C}^\alpha_g$. $\mathbb{Y}^g = \mathbb{Y} + \frac{1}{2}\delta[\mathbf{Y}]$.

i.e. bijection: $\mathcal{C}^\alpha([0,T], V) \leftrightarrow \mathcal{C}^\alpha_g \oplus C^{2\alpha}(\square, \mathrm{Sym}(V \otimes V))$

Rmk: $\mathbb{Y}^g$ is called geometrication of $\mathbf{Y}$.

And notice $\frac{1}{2}\delta[\mathbf{Y}]$ is from Itô-Stratonovich relation

Def: $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{Y} \in \mathcal{C}^\alpha([0,T], \mathbb{R}^d)$ is RP with Lip. bracket (denoted by $\mathbf{Y} \in \mathcal{C}^{\alpha,1}$) if $t \mapsto [\mathbf{Y}]_t$ is Lip. Set $V_t^{\mathbf{Y}} := \frac{d[\mathbf{Y}]_t}{dt} \in S^d$. And denote $\mathbf{Y} \in \mathcal{C}^{\alpha,1+}$ if in addition $V_t^{\mathbf{Y}} > 0$. denote $\sqrt{V^{\mathbf{Y}}}$ is the unique square root of $\sqrt{V^{\mathbf{Y}}}(\sqrt{V^{\mathbf{Y}}})^T = V^{\mathbf{Y}}$. $\sqrt{V^{\mathbf{Y}}} \in S^d_+$.

e.g. Consider $m_t = m_0 + \int_0^t \sigma_s \, dB_s$. where

$\sigma \in \mathbb{R}^{d\times n}$, $\mathcal{F}^B$-progressive. s.t. $\|\sigma\|_{\infty,[0,T]} < \infty$

<u>prop.</u> $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\Rightarrow \mathbb{M}^{It\hat{o}} \in \mathcal{C}^{\alpha}([0,T], \mathbb{R}^d)$

and $\mathbb{M}^{strat} \in \mathcal{C}_g^{\alpha}$. a.s.

<u>Rmk</u>: Note by def of Itô/Strat.

$$\mathbb{M}^{It\hat{o}}_{s,t} = \mathbb{M}^{strat}_{s,t} - \frac{1}{2}\delta[M]_{s,t}$$

$\Rightarrow$ By uniqueness: $[\mathbb{M}^{It\hat{o}}] = [M]$.

And since $[M] = \int \sigma_t \sigma_t^T dt$

So: $d[\mathbb{M}^{It\hat{o}}]_t / dt = \sigma_t \sigma_t^T > 0$.

② <u>Viscosity Sol. to PDEs</u>:

The prime virtues of this theory is that it allows merely Conti. func. to be sol. for 2^{nd}-order PDE and provides existence and uniqueness theory.

<u>Rmk</u>: Regular sol. may not exist and weak sol. may not be unique.

Next consider $F(x, u, Du, D^2u) = 0$. $x \in \Omega \subset \mathbb{R}^n$.

<u>Def</u>: u is viscosity Subsol. to the PDE at x_0 if $\forall$ test func. $\varphi \in C^2(\Omega)$. s.t. $u - \varphi$

has local max. at x_0

$\Rightarrow F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0$.

u is viscosity supersol. if $u-\varphi$ has local min. at x_0

$\Rightarrow F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$.

And if u is viscosity sol. if it's viscosity sub- and super-sol.

Rmk: i) $v \in \mathcal{F}$ is family of vis-subsol. $\Rightarrow$ $\sup_{\mathcal{F}} v(x) =: u(x)$ is vis-subsol.

ii) Viscosity sol. indicates for any smooth func φ. when it's tangent with the sol. $\Rightarrow$ some "rule of sign" for the PDE won't be violated.

iii) It's useful when coping with some optimal problem.

prop. $u \in C^2$ is classical sol. Then u is a viscosity sol. if

i) F doesn't depend on D^2u. or

ii) $F(x, z, p, M) \leq F(x, z, p, N)$. if $M \geq N$

__Rmk__: Conversely, we see if viscosity sol. u is regular. then it's classical sol. (let $\varphi = u$)

__Def__: F is proper if $F(x, z, p, M) \leq F(x, z', p, M)$ for $z \leq z'$.

__Rmk__: It's for comparison principle and uniqueness.

__Prop.__ If F, F_ε conti. s.t. $F_\varepsilon \to F$. and u_ε is viscosity sol. to $F_\varepsilon(\square) = 0$. s.t. $u_\varepsilon \xrightarrow{ucc} u$

Then: u is the viscosity sol.

__eg.__ The viscosity sol. u to

$$\begin{cases} u_t + H(x, Du) = 0 \\ u(0) = g \end{cases}$$ 1^{st}-order Cauchy problem

is limit of classical sol. u^ε to

$$\begin{cases} u^\varepsilon_t + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon. \\ u^\varepsilon(0) = g \end{cases}$$

where we add viscoisty term $\varepsilon \Delta u^\varepsilon$.

<u>Thm.</u> (Perron's method)

Given same bdy condition. If

i) Comparison prin. holds. i.e. u, v are viscosity subsol. supersol. resp. $\Rightarrow u \le v$.

ii) If $\bar{u}, \underline{u}$ viscosity supersol. and subsol. resp. both exist.

Then: $W(x) = \sup\{ w(x) \mid \underline{u} \le w \le \bar{u},\ w$ is vis. subsol.$\}$ is a viscosity sol.

(2) <u>Conditional Laws:</u>

Next assume $f, g \in C_B^3([0,T]\times\mathbb{R}; \mathbb{R})$. $V = v^2$.

$I_t := \int_0^t v_s\,dW_s$. The dynamics is given by

$dX_s^{t,x} = f(s, X_s^{t,x})\,dI_s + g(s, X_s^{t,x})\,v_s\,dB_s$ (*)

<u>Rmk:</u> We have strong well-posedness for it.

Replace $(I_t, \mathbf{I}_t)$ by general deterministic RP pair $(\mathcal{I}, v^{\mathcal{I}})$ where $\mathbf{I}_t \triangleq \int_0^t \delta I_{s,t}\,dI_s$

We have: $dX_s^{t,x,\mathcal{I}} = f(s, X_s^{t,x,\mathcal{I}})\,d\mathcal{I}_s + g(s, X_s^{t,x,\mathcal{I}})\,V_s^{\mathcal{I}}\,dB_s$

<u>Rmk:</u> Here $[\mathbf{I}] = [I] = \int V$. $v^{\mathcal{I}} = |v|$.

So next we hope sol. $X_s^{t,x,\mathcal{I}}|_{\mathcal{I}=I}$ will be the sol. to (*). To ensure its well-posed, we consider form of the RDE with joint lift.

Def: $I \in \mathcal{C}^{\alpha}([0,T], \mathbb{R}^d)$. $\forall$ $\mathcal{F}^B_t$ - local mart. M.

joint lift $\mathbf{Z}^I(w) = (Z^I(w), \mathbb{Z}^I(w))$ where

$Z^I(w) = (M(w), Y)$. $\mathbb{Z}^I(w) = \begin{pmatrix} \int_s^t \delta M dM & \int_s^t \delta M dY \\ \int_s^t \delta Y dM & \mathbb{Y} \end{pmatrix}$

where $\int \delta Y dM$ is Itô

integral. $\int \delta M dY = \delta M Y - \int \delta Y dM$ (IBP)

Thm: $\forall I \in \mathcal{C}^{\frac{1}{2}+}([0,T], \mathbb{R}^{d'})$. $M^I = \int v^I dB \Rightarrow$

joint lift $\mathbf{Z}^I \in \mathcal{C}^{\alpha'}$. a.s. $\forall \alpha' \in (\frac{1}{3}, T)$.

Besides, there exists unique sol. to RDE

$dX^{t,x,I}_s(w) = (g, f)(s, X^{t,x,I}_s(w)) d\mathbf{Z}^I_s(w)$

which is time-inhomo Markovian and a.s.

$\mathcal{L}(X^{t,x} \mid \mathcal{F}^W_T \vee \mathcal{G}^B_t)(w) = \mathcal{L}(X^{t,x,I} \mid \mathcal{G}^W_T \vee \mathcal{G}^B_t)(w) =$

$\mathcal{L}(X^{t,x,Y})|_{Y=I(w)}$. where $X^{t,x}$ is sol. to SDE

(*) and $X^{t,x,I}$ is sol. to RSDE of (*).

Pf: use BDG inequi. and Kolmogorov's

criterin $\Rightarrow \mathbf{Z} \in \mathcal{C}^{\alpha'}$.

And $f, g \in C_B^3 \Rightarrow$ well-posedness.

We also see $X^{t,x,I}$ is rep-limit of

time-inhomo. Markov X^{t,x,Y^ε} ($Y^\varepsilon \to Y$
and Y^ε is piecewise linear RP)

(3) Feynmann-Kac Represent:

Next, we will exploit the Markovian nature of $X^{t,x,Y}$ to the RDE in Thm'.

Let $\varphi \in C_B(\mathbb{R}', \mathbb{R}')$, $Y \in \mathcal{C}^{q,1+}$. Consider RPDE:

$$\begin{cases} -d_t u^Y = L_t(u^Y) V^Y dt + \Gamma_t(u^Y) d Y^g_t, \quad t \in [0,T) \\ u^Y(T,x) = \varphi(x) \end{cases}$$

where
$$\begin{aligned} L_t[u^Y](x) &:= \frac{1}{2}g^2(t,x)\partial^2_{xx}u^Y(t,x) + f_0(t,x)\partial_x u^Y(t,x) \\ \Gamma_t[u^Y](x) &:= f(t,x)\partial_x u^Y(t,x), \quad f_0 = -\frac{1}{2} f \partial_x f. \end{aligned}$$

Rmk: Note f_0 here is from the RDE of $X^{t,x,Y}$ in Thm' is equi. with:

$$dX^{t,x,Y}_s = g(s, X^{t,x,Y}_s)\mathbf{v}^Y_s dB_s + f_0(s, X^{t,x,Y}_s)\mathbf{V}^Y_s ds + f\left(s, X^{t,x,Y}_s\right) d\mathbf{Y}^g_s$$

Here, recall $\exists\, Y^{g,\varepsilon} \in Lip.$ s.t. $Y^{g,\varepsilon} \to Y^g$

So $dY^{g,\varepsilon}_t = \dot{Y}^{g,\varepsilon} dt$. We get classical PDE:

$-\partial_t u = L_t(u) V^Y_t + \Gamma_t(u) \dot{Y}^{g,\varepsilon}_t$. $u(T,x) = \varphi(x)$.

We know $\forall\, C^{1,2}$-sol. u^ε to this PDE has unique represent $u^\varepsilon(t,x) = \mathbb{E}(\varphi(X^{t,x,\varepsilon}_T))$

where $X^{t,x,\epsilon}$ is unique strong sol. to SDE:

$$dX_s^{t,x,\epsilon} = g(s, X_s^{t,x,\epsilon})\mathbf{v}_s^{\mathbf{Y}}\, dB_s + \left(f_0(s, X_s^{t,x,\epsilon})\mathbf{V}_s^{\mathbf{Y}} + f(s, X_s^{t,x,\epsilon})\dot{Y}_s^{\epsilon}\right) ds.$$

Rmk: $C^{1,2}$ for sol. u^ϵ can't be guaranteed so may not be unique while $u^\epsilon(t,x) = \mathbb{E}(q(X_T^{t,x,\epsilon}))$ is weak sol. and the unique viscosity sol. (only require conti.)

Thm: $\mathbb{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R}^1)$, $Y^\epsilon \in Lip$ with lift $\mathbb{Y}^{g,\epsilon}$ st. $\mathbb{Y}^{g,\epsilon} \to \mathbb{Y}^g$ in $\mathscr{C}^\alpha$. And u^ϵ is the unique bdd viscosity FK sol. above. Then $\exists u^{\mathbb{Y}} \in C_B([0,T]\times\mathbb{R};\mathbb{R})$, st. $u^\epsilon \to u^{\mathbb{Y}}$ pointwise with FK repre.: $u^{\mathbb{Y}}(t,x) = \mathbb{E}(q(X_T^{t,x,\mathbb{Y}}))$

Besides $S: \begin{array}{ccc} C_B \times \mathscr{C}^{\alpha,1+} & \to & C_B \\ (q, \mathbb{Y}) & \mapsto & u^{\mathbb{Y}} \end{array}$ is conti.

Pf: i) The RDE of $X^{t,x,\mathbb{Y}}$ is equi. with

$$dX_s^{t,x,\mathbf{Y}} = g(s, X_s^{t,x,\mathbf{Y}})\mathbf{v}_s^{\mathbf{Y}}\, dB_s + f_0(s, X_s^{t,x,\mathbf{Y}})\mathbf{V}_s^{\mathbf{Y}}\, ds + f\left(s, X_s^{t,x,\mathbf{Y}}\right) d\mathbf{Y}_s^g$$

ii) Denote $\mathbb{Y}^\epsilon$ is rough lift of Y^ϵ. i.e. $\mathbb{Y}^\epsilon = \mathbb{Y}^{g,\epsilon} - \frac{1}{2}\delta[\mathbb{Y}]$

$\Rightarrow \|\mathbb{Y}^\epsilon - \mathbb{Y}\|_{\alpha'} \to 0$. $\forall \alpha' \in (\frac{1}{3}, \alpha)$.

iii) $$dX_s^{t,x,\epsilon} = g(s, X_s^{t,x,\epsilon})\mathbf{v}_s^{\mathbf{Y}}\, dB_s + \left(f_0(s, X_s^{t,x,\epsilon})\mathbf{V}_s^{\mathbf{Y}} + f\left(s, X_s^{t,x,\epsilon}\right)\dot{Y}_s^{\epsilon}\right) ds,$$

is well-posed. And we can prove

the sol. $X^{t,x,\varepsilon} \overset{a.s.}{=} X^{t,x,Y^\varepsilon}$ (the sol. to RDE of $X^{t,x,Y}$ in Thm' by replace Y^ε)

So: $X^{t,x,Y^\varepsilon} = X^{t,x,\varepsilon} \xrightarrow{ucp} X^{t,x,Y}$

$\Rightarrow u^\varepsilon \to u^Y := \mathbb{E}(\varphi(X^{\cdot,\cdot,Y}))$ pointwise.

(By some Ascoli argument. we have $u^\varepsilon \xrightarrow{ucc} u^Y$ in fact.)

4) Continuity is from stab. of RDE sol.

<u>Rmk</u>: Replace dY^Z by $\circ dI$ (strat. lift.)

$\Rightarrow$ we have linear SPDE:

$-d_t u = L_t(u)dt + \Gamma(u) \circ dW_t \quad (*')$

(WLOG, $V_t = 1$. or absorb it into L_t)

By using backward Strat. integration against W, we can give meaning to sol. u_t for $(*')$, adapted to

$\mathcal{G}^W_{t,T} = \sigma(W_v - W_u,\ t \le u \le v \le T)$

But it's hard to generalize W to local mart. (semimart is not stable under time-reversal)

<u>Rmk</u>: We consider the SPDE in

Strat. form here because the wellposedness for Itô form require Stoch. parabolic condition

(4) <u>Option pricing</u>:

Restrict on Vanilla options, $\varphi \in C_B(\mathbb{R}^1;\mathbb{R}^1)$

Next, we want to compute $\mathbb{E}(\varphi(X_T^{t,x}))$

To make use of Markov nature of B.

We first consider $u(t,y,w) = \mathbb{E}(\varphi(X_T^{t,x}) \mid \mathcal{F}_T^W)$

<u>Thm</u>[3]. For $g, f \in C_B^3$, $Y \in C^{\alpha,1+}$, $u^Y(t,x)$ is the Sol. to the RPDE in Thm[2]

$\Rightarrow u(t,y,w) = u^Y(t,x)\big|_{Y=I(w)}$. a.s.

<u>Pf</u>: By Thm[1]. $\mathcal{L}(X^{t,x,I} \mid \mathcal{F}_T^W \vee \mathcal{F}_t^B) = \mathcal{L}(X^{t,x} \mid \mathcal{F}_T^W)$

And note $\exists$ Borel-measur. func. F. s.t. $\varphi(X^{t,x,I}) = F(I,B)$.

$\Rightarrow$ LHS $= \mathbb{E}(F(I,B) \mid \mathcal{F}_T^W \vee \mathcal{F}_t^B)$

$\overset{\text{Thm}^1}{=} \int_C F(Y,b)\, \mu_B(db) \big|_{Y=I(w)}$

$= \mathbb{E}(\varphi(X_T^{t,x,Y}))\big|_{Y=I(w)} = u^{I(w)}(t,x)$

where μ_B is r.c.d. of B w.r.t. $\mathcal{F}_T^W$.

Cor. We obtain the price at $t=0$ is

$$y_0 = \mathbb{E}(u^I(0, X_0)).$$

prop. (Price at any time t)

Under same assumpt as above. We have

$\mathbb{E}(g(X_T) \mid \mathcal{F}_t) = \mathbb{E}(u^I(t,x) \mid \mathcal{F}_t^W)\big|_{x=X_t}$. a.s.

Pf. Note $X_s^{t,x}$ satisfies integral eq.:

$$X_s^{t,x} = x + \int_t^s f(u, X_u^{t,x}) V_u dW_s + g(\dots) V_u dB_u$$

and it's strong unique.

$\Rightarrow X_s^{t,x}$ is flow sol. $\Rightarrow X_T = X_T^{t,x}$

Rmk: Note def of flow sol. also requires measurability $(w, x) \mapsto X_t^{s,x}(w)$ is $\mathcal{F}_t \otimes \mathcal{B}_{\mathbb{R}^d}$.

It's true from continuity of SDE sol.: $x \mapsto X_t^{s,x}$ conti.

Note $x \mapsto X^{t,x,I}$ by conti. of RDE sol.

So: $x \mapsto \mathbb{E}(g(X_T^{t,x,I}) \mid \mathcal{F}_T^W \vee \mathcal{F}_t^B)$ is well-def and conti. by DCT.

$\Rightarrow \mathbb{E}(g(X_T) \mid \mathcal{F}_t) = \mathbb{E}(g(X_T^{t,X_t}) \mid \mathcal{F}_t)$

$$\overset{\text{Thm}'}{=} \mathbb{E}\left(\mathbb{E}\left(\phi(X_T^{t,X_t,\mathbf{Y}}) \mid \mathcal{F}_T^W \vee \mathcal{F}_t^B \right) \mid \mathcal{F}_t \right)$$

$$\overset{\text{mp}}{=} \mathbb{E}\left(\mathbb{E}\left(\phi(X_T^{t,x,\mathbf{Y}}) \mid \mathcal{F}_T^W \right) \mid \mathcal{F}_t^W \right) \Big|_{x=X_t}$$

$$\overset{\text{Thm}^3}{=} \mathbb{E}\left(u^{\mathbf{Y}}(t,x) \mid \mathcal{F}_t^W \right) \Big|_{x=X_t} \quad \text{a.s.}$$

<u>eg.</u> i) $f = \rho,\ g = (1-\rho^2)^{1/2},\ \rho \in [-1,1]$.

$\Rightarrow dX_s^{t,x} = V_s\left(\rho\, dW_s + (1-\rho^2)^{1/2} dB_s \right)$

With RPDE
$$\begin{cases} -d_t u^{\mathbf{Y}} & = \frac{1}{2}(1-\rho^2)\partial_{xx}^2 u_t^{\mathbf{Y}} \mathbf{V}_t^{\mathbf{Y}} dt + \rho \partial_x u_t^{\mathbf{Y}} d\mathbf{Y}_t^g \\ u^{\mathbf{Y}}(T,x) & = \phi(x). \end{cases}$$

Choose $Y^{g,\varepsilon}$ (resp. Y^ε) geodesic approxi.

consider
$$\begin{cases} -\partial_t u^\epsilon & = \frac{1}{2}(1-\rho^2)\mathbf{V}_t^{\mathbf{Y}} \partial_{xx}^2 u_t^\epsilon + \rho \dot{Y}_t^\epsilon \partial_x u_t^\epsilon \\ u^\epsilon(T,x) & = \phi(x). \end{cases}$$

Let $u^\varepsilon(t,x) =: V(t, x + \rho\, \delta Y_{t,T}^\varepsilon)$. V will

solve
$$\begin{cases} -\partial_t v & = \frac{1}{2}(1-\rho^2)\mathbf{V}_t^{\mathbf{Y}} \partial_{xx}^2 v_t \\ v(T,x) & = \phi(x). \end{cases}$$

In fact, we can find $V = \phi * f$.

where
$$f(t,x,y) := \frac{1}{\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(y-x)^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\}.$$

So the sol. u^ε is expressed in:

$$u^\epsilon(t,x) := v(t, x + \rho Y_{t,T}^\epsilon) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(y-x-\rho Y_{t,T}^\epsilon)^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy.$$

Using dominated convergence, we finally find the solution to the RPDE is explicitely given by

$$u^{\mathbf{Y}}(t,x) = \lim_{\epsilon \to 0} u^\epsilon(t,x) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(y-x-\rho Y_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy.$$

Rmk: V just satisfies heat equation with extra multiple V^Y. recall $[I]=\int V ds$

$\Rightarrow$ We get expression of f.

So by randomizing $Y = I(\omega)$, we have:

$$u(t,x,\omega) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi(1-\rho^2)[I]_{t,T}(\omega)}} \exp\left\{-\frac{(y-x-\rho I_{t,T}(\omega))^2}{2(1-\rho^2)[I]_{t,T}(\omega)}\right\} dy.$$

Rmk: Note $X_T^{t,x} = x + \rho\int_t^T V_s dW_s + \square \sim$

$N(x+\rho\,\delta I_{t,T}, (1-\rho^2)\,\delta[I]_{t,T})\,|\,W \Rightarrow u(t,x,\omega)$

ii) $f = \rho x$, $g = (1-\rho^2)^{1/2} x$, $\rho \in [-1,1]$.

$\Rightarrow dX_s^{t,x} = X_s^{t,x} V_s(\rho\, dW_s + (1-\rho^2)^{1/2} dB_s)$

with
$$\begin{cases} -d_t u^{\mathbf{Y}} &= \left(\frac{1}{2}x^2(1-\rho^2)\partial^2_{xx}u^{\mathbf{Y}}_t - \frac{1}{2}\rho^2 x\partial_x u^{\mathbf{Y}}_t\right)\mathbf{V}^{\mathbf{Y}}_t dt + \rho x \partial_x u^{\mathbf{Y}}_t\, d\mathbf{Y}^g_t \\ u^{\mathbf{Y}}(T,x) &= \phi(x). \end{cases}$$

Consider ϵ-approxi. PDE:

$$\begin{cases} -\partial_t u^\epsilon &= \frac{1}{2}x^2(1-\rho^2)\mathbf{V}^{\mathbf{Y}}_t\partial^2_{xx}u^\epsilon_t + \left(\rho x \dot{Y}^\epsilon_t - \frac{1}{2}\rho^2 x \mathbf{V}^{\mathbf{Y}}_t\right)\partial_x u^\epsilon_t \\ u^\epsilon(T,x) &= \phi(x). \end{cases}$$

We can reduce that PDE to

$$\begin{cases} -\partial_t v &= \frac{1}{2}x^2(1-\rho^2)\mathbf{V}^{\mathbf{Y}}_t\partial^2_{xx}v_t \\ v(T,x) &= \phi(x). \end{cases}$$

by letting $u^\epsilon(t,x) =: v(t, x e^{\psi^\epsilon(t)})$, where

$\psi^\epsilon(t) = \rho\,\delta Y^\epsilon_{t,T} - \frac{1}{2}\rho^2\,\delta[Y]_{t,T}$

We can solve $v(t,x)$, given by:

$$v(t,x) = \int_{\mathbb{R}_+} \phi(y) \frac{1}{y\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(\ln(y/x) - \frac{1}{2}(1-\rho^2)[\mathbf{Y}]_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy.$$

Using dominated convergence, and applying Theorem 3.5, we find in this case

$$\begin{aligned} u^{\mathbf{Y}}(t,x) &= \lim_{\epsilon\to 0} v(t, xe^{\psi^\epsilon(t)}) \\ &= \int_{\mathbb{R}_+} \phi(y) \frac{1}{y\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(\ln(y/x) - \rho Y_{t,T} + \frac{1}{2}[\mathbf{Y}]_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy. \end{aligned}$$

So:

$$u(t,x,\omega) = \int_{\mathbb{R}_+} \phi(y) \frac{1}{y\sqrt{2\pi(1-\rho^2)[I]_{t,T}(\omega)}} \exp\left\{ -\frac{(\ln(y/x) - \rho I_{t,T}(\omega) + \frac{1}{2}[I]_{t,T}(\omega))^2}{2(1-\rho^2)[I]_{t,T}(\omega)} \right\} dy.$$

Rmk: i) The formulas can be deduced from Romano-Touzi formula.

ii) Similarly, $X_T^{t,x}|_W$ has log-normal distribution.

Procedure to derive u:

i) Write down RPDE of u.

ii) Choose approxi: $\mathbf{Y}^\varepsilon$. Write down ε-PDE for u^ε.

iii) Solve u^ε. Let $\varepsilon \to 0$ to obtain $u^{\mathbf{Y}}$.

iv) Randomize $u^{\mathbf{Y}}|_{\mathbf{Y}=\mathbf{I}(\omega)}$.

(5) Multivariate Case:

We can consider d_x-dim LSV model

$dX_s^{t,x} = F(s, X_s^{t,x}) v_s \, dW_s + G(s, X_s^{t,x}) v_s \, dB_s$. Where

$F, G : [0,T] \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d}$ smooth. And B, W are indept d-dim BMs, $V_t \in \mathbb{R}^{d \times d}$ satisfy assumpt.

<u>Rmk</u>: B and W can be in different dim.

by letting $V \in \mathbb{R}^{d_1 \times d_2}$ (non-square) and

introducing linear proj. map Π^W, Π^B.

Then Thm¹, Thm² will still hold with

RPDE:
$$\begin{cases}-d_t u &= L_t[u]dt + \Gamma_t[u]d\mathbf{Y}_t^g \quad t \in [0,T[\times \mathbb{R}^{d_X}, \\ u(T,x,\cdot) &= \phi(x), \quad x \in \mathbb{R}^{d_X},\end{cases}$$

where

$$\begin{aligned} L_t[u](x) &= \frac{1}{2}\mathrm{Tr}\left[G(t,x)(\mathbf{v}_t^{\mathrm{Y}})(\mathbf{v}_t^{\mathrm{Y}})^T G(t,x)^T \nabla_{xx}^2 u(t,x)\right] + \langle F_0(t,x), \nabla_x u(t,x)\rangle \\ \Gamma_t[u](x) &= \langle F(t,x), \nabla_x u(t,x)\rangle, \end{aligned}$$

where we define $F_0(t,x) \in \mathbb{R}^{d_X}$ by $F_0^i(t,x) := -\frac{1}{2}\sum_{j,k,l} \partial_{x_k} F^{ij}(t,x) F^{kl}(t,x) (\mathbf{v}_t^{\mathrm{Y}}(\mathbf{v}_t^{\mathrm{Y}})^T)^{lj}$

Similarly, $\mathbb{E}(\phi(X_T^{t,x}) \mid \mathcal{F}_T^W)(\omega) \overset{a.s.}{=} u^{\mathbf{Y}}(t,x)|_{\mathbf{Y}=\mathbf{I}(\omega)}$

where $u^{\mathbf{Y}}$ is unique sol. to RPDE above.

(6) <u>Numerics</u>:

Under the same assumpt. Consider SDE:

$dX_s^{t,x} = f(s, X_s^{t,x}) v_s dW_s + g(s, X_s^{t,x}) v_s dB_s$.

Recall $\mathbb{E}(\phi(X_T^{t,x}) \mid \mathcal{F}_T^W)(\omega) \overset{a.s.}{=} u^{\mathbf{Y}}(t,x)|_{\mathbf{Y}=\mathbf{I}(\omega)}$

Since we can generate Sample $(\mathbf{I}^{(m)})_m$

Next, we solve RPDE for $u^{\mathbf{Y}}$ by FDM.

(Then we have $u^{\mathbf{I}^{(m)}}(t,x) = \mathbb{E}(\phi(X^{t,x,\mathbf{I}^{(m)}}))$)

$\Rightarrow y_0(x) = \mathbb{E}(u^{\mathbf{I}}(0,x)) \approx \frac{1}{M}\sum_1^M u^{\mathbf{I}^{(m)}}(0,x) =: y_0^M(x)$)

Rmk: i) Here $u^{\mathcal{I}^{(m)}}$ is not the option price for $\mathbb{Q}^0$-price $X^{t,x,\mathcal{I}^{(m)}}$ since $X^{t,x,\mathcal{I}^{(m)}}$ isn't $\mathbb{Q}^0$-mart. (has drift from W)

ii) The scheme is more stable than the plain MC, since the conditioning can regularize the payoff func. ϕ.

Consider on $[0,T]\times[a,b]$. $\Delta x = \frac{b-a}{N}$. $\Delta t = \frac{T}{J}$.

Choose Dirichelet bdy condition. We solve

$$
\begin{aligned}
-d_t u^{\mathbf{Y}} &= L_t[u^{\mathbf{Y}}]\mathbf{V}_t^{\mathbf{Y}}dt + \Gamma_t[u^{\mathbf{Y}}]d\mathbf{Y}_t^g, \quad (t,x)\in[0,T[\times]a,b[\\
u^{\mathbf{Y}}(t,a) &= \psi_a(t), \quad t\in[0,T[\\
u^{\mathbf{Y}}(t,b) &= \psi_b(t), \quad t\in[0,T[\\
u^{\mathbf{Y}}(T,x) &= \phi(x), \quad x\in[a,b].
\end{aligned}
$$

① First order FDM:

Here we approxi. $u^{\mathcal{I}_\varepsilon}$ for Σ-PDE w.r.t. $\mathcal{I}_\varepsilon$

Rmk: So $|\bar{u}-u^{\mathbf{Y}}| \le |\bar{u}-u^{\mathcal{I}_\varepsilon}| + |u^{\mathcal{I}_\varepsilon}-u^{\mathbf{Y}}|$

Set $u_j^n = u^{\mathcal{I}^\varepsilon}(t_j, x_n)$, we have on $[t_j, t_{j+1}]$:

$$
u_j^n = u_{j+1}^n + \int_{t_j}^{t_{j+1}} L_s[u^\epsilon](x_n)d[\mathbf{Y}]_s + \int_{t_j}^{t_{j+1}} \Gamma_s[u^\epsilon](x_n)dY_s^\epsilon. \tag{30}
$$

We replace the space derivatives in the differential operators L and Γ by finite difference quotients

$$
\begin{aligned}
L_{t_j}[u^\epsilon](x_n) &\approx L_j^n := \frac{1}{2}g^2(t_j,x_n)\frac{u_j^{n+1}+u_j^{n-1}-2u_j^n}{(\Delta x)^2} + f_0(t_j,x_n)\frac{u_j^{n+1}-u_j^n}{\Delta x} \\
\Gamma_{t_j}[u^\epsilon](x_n) &\approx \Gamma_j^n := f(t_j,x_n)\frac{u_j^{n+1}-u_j^n}{\Delta x}.
\end{aligned}
$$

While recalling to keep the stability. We require $\Delta t/\Delta x^2$ is small in explicit scheme. To improve it. We can use implicit-explicit (IMEX) scheme. i.e. chose left pt for 1^{st} integral and right pt for the 2^{nd} one:

$$u_j^n = u_{j+1}^n + L_j^n[\mathbf{Y}]_{t_j,t_{j+1}} + \Gamma_{j+1}^n Y^{\epsilon}_{t_j,t_{j+1}},$$

for $0 \le j \le J-1, 1 \le n \le N-1$, with boundary conditions

$$u_j^0 = \psi_a(t_j), \quad u_j^N = \psi_b(t_j), \quad u_J^n = \phi(x_n).$$

3 Second order FDM:

We focus on sol. u^Y to the RPDE here. But rather considering the sol. defined by limit of viscosity sol.'s. We consider $u^Y \in C^{0,2}$ is regular sol. and satisfies

$$u^Y(t,x) = \varphi(x) + \int_t^T L_s(u^Y) V_s \, ds + \int_t^T \Gamma_s(u^Y) \, d\mathbf{Y}_s^g$$

with $(\Gamma(u^Y), (\Gamma(u^Y))' = -\Gamma(\Gamma(u^Y))) \in D_Y^{2\gamma}$.

Rmk: Note here $(\Gamma(u^Y))' = \Gamma'(u^Y) + \Gamma((u^Y)')$

$(u^Y)' = -\Gamma(u^Y)$ and $\Gamma' u = 0$. since $f \in C_b^3$

So Gubnelli derivative of f w.r.t. Y is 0.

Note $Y \in \mathcal{K}' \Rightarrow \mathbb{Y}^g_{s,t} = \frac{1}{2}(\delta Y_{s,t})^2$

$$S_1 : \int_t^T \Gamma_s(u^Y)\, d\mathbf{Y}^g_s$$

$$= \lim_{|\pi| \to 0} \sum \left(\Gamma_s(u^Y)\delta Y_{r,s} + \frac{1}{2}\Gamma_s(\Gamma(u^Y))(\delta Y_{r,s})^2 \right)$$

from backward representation for RI.

For $$u^{\mathbf{Y}}(t_j, x_n) = u^{\mathbf{Y}}(t_{j+1}, x_n) + \int_{t_j}^{t_{j+1}} L_s[u^{\mathbf{Y}}](x_n)\mathbf{V}^{\mathbf{Y}}_s\, ds + \int_{t_j}^{t_{j+1}} \Gamma_s[u^{\mathbf{Y}}](x_n) d\mathbf{Y}^g_s.$$

Denoting $u^n_j := u^{\mathbf{Y}}(t_j, x_n)$ for $0 \le j \le J$ and $0 \le n \le N$, we use the same approximations for L and Γ given in the last section, and define

$$(\Gamma'_{t_j}[u])(x_n) \approx (\Gamma')^n_j := -2f_0(t_j, x_n)\frac{u^{n+1}_j - u^n_j}{\Delta x} + f^2(t_j, x_n)\frac{u^{n+1}_j + u^{n-1}_j - 2u^n_j}{(\Delta x)^2}.$$

We still apply the IMEX scheme:

$$u^n_j = u^n_{j+1} + L^n_j[\mathbf{Y}]_{t_j,t_{j+1}} + \Gamma^n_{j+1} Y_{t_j,t_{j+1}} + \frac{1}{2}(\Gamma')^n_{j+1} Y^2_{t_j,t_{j+1}}, \quad 0 \le j \le J-1,\ 1 \le n \le N-1.$$