General Definitions

(1) Discrete RWs:

Def: i) $V = \{x_1, x_2 \cdots x_l\} \subset \mathbb{Z}^d / \{0\}$ is generating set if $\forall y \in \mathbb{Z}^d$. $\exists (k_i)$. s.t. $y = \sum_1^l k_i x_i$.

ii) $\mathcal{G} = \{ V$ generating set $| \ \forall x \in V$. the first nonzero component of $x > 0 \}$

iii) Symmetric. irred. RW is given by:

$V \in \mathcal{G}$ and $k : V \to [0,1]$. s.t.

$\sum_1^l k(x_i) \le 1$. for $V = \{x_i\}_1^l$.

associated with $\mathbb{P}$. p.m on $\mathbb{Z}^d$. by:

$\mathbb{P}(x_i) = \mathbb{P}(-x_i) = k(x_i)/2$. $\mathbb{P}(0) = 1 - \sum_1^l k(x_i)$

iv) For such p. we call it a increment dist. for the walk.

v) Set $\mathcal{P}_d$ is set of such $\mathbb{P}$ on $\mathbb{Z}^d$.

$\mathcal{P} = \bigcup_{d \ge 1} \mathcal{P}_d$. set of symmetric RWs

consider time-homogeneous random walk $S_n =: S_0 + X_1 + X_2 + \cdots X_n$. $\mathbb{P}(S_n = y \mid S_0 = x) =: P_n(x, y)$.

Rmk: i) $P_{2n}(0,0) \geq P_{2n}(0,x)$. $\forall x \in \mathbb{Z}^d$.

Pf: $LHS = \sum_{\mathbb{Z}^d} P_n^2(y)$.

$$\overset{\text{Am-Gm.}}{\geq} \sum_{\mathbb{Z}^d} P_n(y) P_n(y+x) = RHS$$

ii) $P_{2n}(0) \geq (P_2(0))^n \geq p(x)^{2n} > 0$.

for some $x \in \mathbb{Z}^d$.

Def: i) A random walk is bipartite if $P_n(0,0) = 0$. for $\forall$ odd n. it's aperiodic otherwise.

ii) If p is bipartite. partition $\mathbb{Z}^d = (\mathbb{Z}^d)_e \cup (\mathbb{Z}^d)_o$. the set can be reached by even / odd steps.

Rmk: $(\mathbb{Z}^d)_e$ is additive subgroup of $\mathbb{Z}^d$. of index 2.

$(\mathbb{Z}^d)_o$ is the coset.

iii) For random walk $X_n = (X_n^1, \cdots X_n^d)$

Define covariance matrix $\Gamma = (\mathbb{E}(X_1^i X_1^j))$

Rmk: i) Γ is sym. $\Gamma > 0$. can be decomposed in $\Gamma = \Lambda \Lambda^T$. $\Lambda \geq 0$.

iii) If $\Gamma \sim \operatorname{diag}\{\sigma_1^2 \cdots \sigma_d^2\}$. Then.

$\exists$ o.n.b $(u_i)_1^d$ of $\mathbb{R}^d$. s.t.

$$\Gamma x = \sum_1^d \sigma_j^2 (x^T \cdot u_j) u_j.$$

iv) $\mathcal{J}^*(x)^2 = x^T \Gamma^{-1} x$. for $x \in \mathbb{R}^d$. $\mathcal{J}(x)^2 = \mathcal{J}^*(x)^2/d$.

<u>Rmk:</u> $\mathbb{E}(\mathcal{J}(X_1)^2) = \frac{1}{d}\mathbb{E}(|\Lambda^{-1}X_1|^2) = 1.$

<u>Prop.</u> (Construction from SRW)

For $V = \{x_1 \cdots x_l\} \in \mathcal{G}$. $k : V \to [0,1]$

satisfies $\sum_1^l k(x_i) \leq 1$. Then $\exists$ random

walk S_n corresponds to them.

<u>Pf:</u> Suppose $(S_{n,i})_{i=1}^l$ is l i.i.d one

dimensional SRW. with $L_n = (L_n^1 \cdots$

$L_n^l)$ is indept multinomial process

of prob. $(k(x_1) \cdots k(x_l))$. Then.

set: $S_n = \sum_1^l x_i \cdot S_{L_n^i, i}$

<u>Rmk:</u> The idea is just split the decision

of how to jump n times into:

1) Choose x_k from $\{x_i\}_1^l$

2) Decide move by $-x_k$ or $+x_k$.

prop. For $p \in \mathcal{P}_d$

i) If $j_k \geq 0$. $1 \leq k \leq d$. s.t. $\sum_1^d j_k$ is odd.

Then : $\mathbb{E}((X_1^1)^{j_1} \cdots (X_1^d)^{j_d}) = 0$.

ii) $\mathcal{J} \sim \mathcal{J}^* \sim |\cdot|_d$ are equi. norms.

Def: i) $\mathcal{P}_d^* = \{ p$ on $\mathbb{Z}^d \mid \forall x, y \in \mathbb{Z}^d$. $\exists N$. s.t.

$p_n(x, y) > 0$ for $\forall n > N \}$.

set of aperiodic irreducible RWs.

ii) $\mathcal{P}_d' = \{ p \in \mathcal{P}_d^* \mid p$ has zero mean and

finite second moment $\}$.

Denote $\mathcal{P}^* = \cup \mathcal{P}_d^*$. $\mathcal{P}' = \cup \mathcal{P}_d'$

Rmk: Although $\forall p \in \mathcal{P}$ has finite moments.

$\mathcal{P} \not\subset \mathcal{P}'$. Since $\mathcal{P}$ contains bipartite walks

But if $p \in \mathcal{P}$ aperiodic. Then $p \in \mathcal{P}'$.

(2) Conti. Case:

Def: Given V. k. p. conti-time RW with

increment dist. p is conti-time Markov

chain $\tilde{S}_t$ with rates p. i.e.

$\mathbb{P}(\tilde{S}_{t+\Delta t} = y \mid \tilde{S}_t = x) = p(y-x)\Delta t + o(\Delta t)$. $y \neq x$.

$\mathbb{P}(\tilde{S}_{t+\Delta t} = x \mid \tilde{S}_t = x) = 1 - \sum_{y \neq x} p(y-x)\Delta t + o(\Delta t)$

Rmk: From above, we obtain KB formula:

$$\frac{d}{dt}\tilde{p}_t(x) = \sum_{\mathbb{Z}^d} p(y)\left(\tilde{p}_t(x-y) - \tilde{p}_t(x)\right)$$

prop. (construction from S_n)

S_n is discrete-time RW with increment dist. p and $N_t \sim \mathrm{POI}(t)$, indept. r.v.

$\Rightarrow S_{N_t}$ is conti-time RW with increment dist. p.

prop. (construction from RW)

For $p \in \mathcal{P}_d$, $V = \{x_1, \cdots x_l\}$. If $(\tilde{S}_{t,i})_{i=1}^{l}$ is seq of indept one-dim conti-time RW with increment dist. (q_i), $q_i(\pm 1) = p(x_i)$.

Then $\tilde{S}_t =: \sum_i^{l} x_i \cdot \tilde{S}_{t,i}$ is CTRW with p.

Rmk: Note that the coordinates of CTRW (under directions $\{x_i\}_1^{l}$) are indept.

But it fails in discrete case.

e.g. $(\tilde{S}_{t,i})_{t\geq 0}$, $1\leq i\leq d$ are indept 1-dim CT-SRW $\Rightarrow \tilde{S}_t = (\tilde{S}_{t/d,1}, \tilde{S}_{t/d,2}, \cdots \tilde{S}_{t/d,d})$ is CT-SRW in $\mathbb{Z}^d$.

(3) <u>On other lattices:</u>

<u>Def:</u> A lattice $\mathbb{L}$ is discrete additive subgroup of $\mathbb{R}^d$. (Discrete: $\exists\, U \subseteq \mathbb{R}^d$. nbd of 0. s.t. $\mathbb{L} \cap U = \{0\}$. e.g. $\mathbb{L} =: \{k_1\sqrt{2} + k_2 \cdot 2 \mid k_1, k_2 \in \mathbb{Z}\}$ isn't discrete)

<u>prop.</u> If $\mathbb{L}$ is lattice in $\mathbb{R}^d$. Then $\exists\, k \le d$. and $\{x_i\}_1^k \in \mathbb{L}$. l.i. vectors in $\mathbb{R}^d$. s.t. $\mathbb{L} = \{\sum_1^k j_i x_i \mid j_i \in \mathbb{Z}.\ 1 \le i \le k\}$.

<u>Rmk:</u> If $k \le d$. $\mathbb{L}$ is k-dim lattice in $\mathbb{R}^d$. Then $\exists\, A : \mathbb{R}^d \to \mathbb{R}^k$, a linear operator. which is isomorphism of $\mathbb{L}$ onto $\mathbb{Z}^k$.

We call $|\det A|$ is density of $\mathbb{L}$.

$\Rightarrow$ We can only consider the RW running on $\mathbb{Z}^k$.

(4) <u>Generators:</u>

<u>Def:</u> $\mathcal{L} f(x) = \mathbb{E}^x(f(S_1)) - f(x) = \frac{\partial}{\partial t} \mathbb{E}^x(f(\tilde{S}_t))\big|_{t=0}$

is generator for symmetric RW S_n and $\tilde{S}_t$.

prop. For $\hat{\mathcal{L}} f(y) = \frac{1}{2} \sum_{x \in V} k(x) |x|^2 \partial^2_{x/|x|} f(y)$.

i) If $(u_i)_1^d$ is o.n.b. s.t. $\Gamma x = \sum_1^d \sigma_j^2 (x \cdot u_j) u_j$

Then $\hat{\mathcal{L}} f(y) = \frac{1}{2} \sum_1^d \sigma_j^2 \partial^2_{u_j} f(y)$.

ii) If $f \in C^4(\mathbb{R}^d, \mathbb{R}^1)$. Then $\exists c$. s.t. $\forall y \in \mathbb{Z}^d$.

$|\mathcal{L} f(y) - \hat{\mathcal{L}} f(y)| \le c R^4 \cdot M$. $R = \max\{|x| \mid p(x) > 0\}$.

$M = \max_{|x-y| \le R} |\partial^4 / \partial x f|$.

(5) Markov property:

Def: i) For filtration $(\mathcal{F}_k)_0^\infty \uparrow$. $p \in \mathcal{P}_d$. S_n is RW with increment dist. p w.r.t $(\mathcal{F}_k)$

if $\begin{cases} S_n \in \mathcal{F}_n. \ \forall n. \\ S_n - S_{n-1} \text{ indept of } \mathcal{F}_{n-1}. \ \mathbb{P}(\Delta S_n = x) = p(x). \end{cases}$

ii) For filtration $(\mathcal{F}_t)_{t \ge 0} \uparrow$. right-conti. and $p \in \mathcal{P}_d$. $\tilde{S}_t$ is conti-time RW with increment dist. p. w.r.t. $(\mathcal{F}_t)$

if $\begin{cases} \tilde{S}_t \in \mathcal{F}_t. \ \forall t. \\ \tilde{S}_t - \tilde{S}_s \text{ indept of } \mathcal{F}_s. \ \mathbb{P}(\Delta_s^t \tilde{S} = x) = \tilde{p}_{t-s}(x) \end{cases}$

Thm. For S_n. $\tilde{S}_t$ defined as above. τ. $\tilde{\tau}$ are stopping time w.r.t. S_n. $\tilde{S}_t$. Then:

i) On $\{\tau < \infty\}$. $Y_n = S_{n+\tau} - S_\tau$ is RW

with increment p indept of $\mathcal{G}_\tau$.

ii) On $\{\tilde{\tau} < \infty\}$. $\tilde{Y}_t = S_{t+\tilde{\tau}} - S_{\tilde{\tau}}$ is conti-

time RW with increment p indept of $\mathcal{F}_{\tilde{\tau}}$.

Pf: i) is trivial. ii) : Def $\tau_n = \frac{[2^n \tau]}{2^n} \uparrow \tau$.

prop. (Reflection principle)

For S_n, $\tilde{S}_t$. RW with $p \in \mathcal{P}_n$. Start at 0.

i) For $u \in \mathbb{R}^d$. unit. $b > 0$. Then:

$$\mathbb{P}(\max_{j \leq n} S_j \cdot u \geq b) \leq 2\,\mathbb{P}(S_n \cdot u \geq b)$$

$$\mathbb{P}(\max_{s \leq t} \tilde{S}_s \cdot u \geq b) \leq 2\,\mathbb{P}(\tilde{S}_t \cdot u \geq b)$$

ii) $$\mathbb{P}(\max_{j \leq n} |S_j| \geq b) \leq 2\,\mathbb{P}(|S_n| \geq b). \quad b > 0$$

$$\mathbb{P}(\max_{s \leq t} |\tilde{S}_s| \geq b) \leq 2\,\mathbb{P}(|\tilde{S}_t| \geq b). \quad b > 0$$

pf: i) Consider conti. case: $A_n = \{\max_{j \leq 2^n} \tilde{S}_{\frac{j}{2^n}t} \cdot u \geq b\} \uparrow \{\max_{s \leq t} \tilde{S}_s \cdot u \geq b\}$. a.s.

Set $\tau = \inf\{j \mid \tilde{S}_{jt/2^n} \cdot u \geq b\}$.

$$\mathbb{P}(\tilde{S}_t \cdot u \geq b) \geq \sum_1^{2^n} \mathbb{P}(\tau = j,\ (\tilde{S}_t - \tilde{S}_{jt/2^n}) \cdot u \geq 0)$$

$$\overset{\text{mp.}}{\underset{\text{sym}}{\geq}} \frac{1}{2} \sum_1^{2^n} \mathbb{P}(\tau = j)$$

$$= \frac{1}{2}\,\mathbb{P}(A_n).$$

Then set $n \to \infty$.

ii) Similarly. Note that:

Set $\tau = \inf\{ j \geq 0 \mid |\tilde{S}_{jt/2^n}| \geq b \}$.

$\bigcup_{1}^{2^n} \{ \tau = j,\ (\tilde{S}_t - \tilde{S}_{tj/2^n}) \tilde{S}_{tj/2^n} \geq 0 \}$

$\subset \{ |\tilde{S}_t| \geq b \}$.

<u>Prop.</u> (Martingales)

i) $(X_k) \subset \mathbb{R}^d$ indept with zero mean and cov matrix I. Then:

$|S_n|^2 - tr(I) \cdot n$ is mart w.r.t. $\sigma(X_k, k \leq n)$

ii) If S_n is RW with $p \in \mathcal{P}_d \cup \mathcal{P}_d'$ and cov. matrix. I. Then:

$\mathcal{J}(S_n)^2 - n$ is mart. w.r.t. $\sigma(X_k, k \leq n)$

<u>Pf:</u> easy to check.