Local CLT.

<u>Motive:</u> By CLT: $\lim_{n\to\infty} \mathbb{P}(r \le \frac{\sum_1^n X_k}{\sqrt{n}} \le s)$

$$= \int_r^s \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2} dy.$$

for (X_k) i.i.d. zero mean. and var σ^2.

$\Rightarrow$ For S_n RW with $p \in \mathcal{P}_1$. aperiodic.

$p_n(k) \sim e^{-k^2/2\sigma^2 n} / \sqrt{2\pi n\sigma^2}$. When

bipartite: $p_n(k) + p_n(k+1) \approx \int_{k/\sqrt{n}}^{k+2/\sqrt{n}} \square$

$\sim 2e^{-k^2/2n\sigma^2} / \sqrt{2\pi\sigma^2 n}$. $\forall k \in \mathbb{Z}^1$.

<u>Rmk</u>: local CLT gives more precise approxi.

(1) <u>Discrete Case:</u>

① <u>Def</u>: For $p \in \mathcal{P}_d$. with cov matrix. Γ. Set:

$$\bar{p}_n(x) = e^{-J^*(x)^2/2n} / (2\pi n)^{\frac{d}{2}} \sqrt{|\Gamma|}.$$

<u>Thm</u>. (Local CLT).

If $p \in \mathcal{P}_d$. aperiodic. Then $\exists c$. for $\forall k \ge 4$

$\exists \tilde{c}(k) < \infty$. St. $\forall n \in \mathbb{Z}^+$. $x \in \mathbb{Z}^d$. Set $z = \frac{x}{\sqrt{n}}$

We have: $|p_n(x) - \bar{p}_n(x)| \le c / n^{\frac{d+2}{2}} |z|^2$.

and $\le \tilde{c}(k) \left((|z|^k + 1) e^{-\frac{J^*(z)^2}{2}} + n^{-\frac{(k-3)}{2}} \right) / n^{\frac{d+2}{2}}$

<u>Rmk</u>: i) When $k = 4$. for $|x| \le \sqrt{n}$.

Note $\bar{p}_n(x) \sim n^{-\frac{d}{2}} \Rightarrow p_n(x) = \bar{p}_n(x)(1 + O(n^{-1}))$

ii) When $|x| \ge \sqrt{n}$. Note $\bar{p}_n(x) \sim n^{-\frac{d}{2}} e^{-J^*(x)^2/2n}$

$\Rightarrow p_n(x) = \bar{p}_n(x) \left(1 + \frac{O_k(|z|^k)}{n}\right) + O_k\left(n^{-\frac{d+k-1}{2}}\right)$.

<u>prop.</u> (Tail estimate)

i) If $p \in \mathcal{P}_d'$. Start at 0. $k \in \mathbb{Z}^+$. $\mathbb{E}(|X_1|^{2k}) < \infty$.

Then $\exists c < \infty$. s.t. $\mathbb{P}(\max_{j \le n} |S_j| \ge s\sqrt{n}) \le c \cdot s^{-2k}$. $\forall s > 0$.

ii) If $p \in \mathcal{P}_d$. Start at 0. Then $\exists \beta > 0$. $c < \infty$.

s.t. $\mathbb{P}(\max_{j \le n} |S_j| \ge s\sqrt{n}) \le c e^{-\beta s^2}$. $\forall s > 0$.

<u>Thm.</u> (Local CLT for bipartite RW)

If $p \in \mathcal{P}_d$ (bipartite). Then. $\forall k \ge 4$. $\exists c(k) < \infty$.

s.t. $\forall x \in \mathbb{Z}^d$. $n \in \mathbb{Z}^+$. Set $z = x/\sqrt{n}$.

$|p_n(x) + p_{n+1}(x) - 2\bar{p}_n(x)| \le c(k) \left((|z|^k + 1) e^{-J^*(z)^2/2} + n^{-\frac{k-3}{2}} \right) / n^{\frac{d+2}{2}}$

<u>Pf</u>: Set $S_n^* = S_{2n}$ is aperiodic on $(\mathbb{Z}^d)_e$.

Then map $(\mathbb{Z}^d)_e$ to $(\mathbb{Z}^d)$.

we have estimate of $p_n^*(x) = p_{2n}(x)$

Approxi. of p_{2n+1} from: $p_{2n+1}(x) = \sum_{y} p_{2n}(x-y)\,p(y)$

Thm. (Exponential moments)

If $p \in \mathcal{P}_d'$. st. $\exists b>0$. $\mathbb{E}(e^{b|X_1|})<\infty$.

Then $\exists \epsilon>0$. st. $\forall n \in \mathbb{Z}^+$. $x \in \mathbb{Z}^d$. $|x|<n\epsilon$

$$p_n(x) = \bar{p}_n(x)\, e^{O(n^{-\frac{1}{2}} + |x|^3/n^2)}.$$

③ Corollaries:

prop. If $p \in \mathcal{P}'$ with bdd support. Then:

$$\exists c>0. \quad \sum_z |p_n(z) - p_n(z+y)| \le c|y|/n^{\frac{1}{2}}$$

Pf: $LHS = \sum_{|z|\ge n^{\frac{1}{2}+\delta}} + \sum_{|z|\le n^{\frac{1}{2}+\delta}} \square =: A+B$

$$A \le \sum_{\square} p_n(z) + p_n(z+y) = O(n^{-\frac{1}{2}})$$

Estimate of B is from:

$$\nabla_y p_n(z) = \nabla_y \bar{p}_n(z) + O(n^{-\frac{d+2}{2}})$$

Cor. If $p \in \mathcal{P}^*$. Then. $\exists c>0$. We have:

$$\sum_z |p_n(z) - p_n(z+y)| \le c|y|/n^{\frac{1}{2}}$$

Pf: It follows from a decompo. Lemma.

<u>Lemma.</u> For $p \in \mathcal{P}_d^*$. There exists $\varepsilon > 0$.

$q \in \mathcal{P}_d'$ with finite supp. $q' \in \mathcal{P}_d^*$

s.t. $p = \varepsilon q + (1-\varepsilon) q'$.

<u>prop.</u> (Coupling)

If $p \in \mathcal{P}_d^*$. Then $\exists c < \infty$. s.t. if

S_n. S_n^* are RW with p. and

start at x, y respectively we have:

$$\mathbb{P}(S_m \neq S_m^*. \ \exists m \geq n) \leq \frac{c|x-y|}{n}$$

<u>Rmk:</u> For $p. q \in \mathcal{P}_d \cup \mathcal{P}_d'$. s.t. $\frac{1}{2}\sum |p(z) - q(z)| = \varepsilon$

Then we can define r.v. X, Y on the

same prob. space. s.t. $\mathbb{P}(X \neq Y) = \varepsilon$.

<u>e.g.</u>

$$\text{set } \mu(x,y) = \begin{cases} \varepsilon^{-1}(p(x) - f(x))(q(y) - f(y)) & \text{if } x \neq y. \\ f(x). & \text{if } x = y. \end{cases}$$

where $f(z) = \min\{p(z), q(z)\}$.

<u>prop.</u> If $p \in \mathcal{P}_d^*$. Then $\exists c < \infty$. for $\forall n. x$.

$p_n(x) \leq c / n^{\frac{d}{2}}$.

Pf: If $p \in \mathcal{P}_d'$ with bdd supp.

Then the conclusion holds by the first. prop.

If $p \in \mathcal{P}_d^*$. $\Rightarrow$ Apply Accomp. Lemma.

Cor. If $p \in \mathcal{P}_d^*$. Then $\exists c$. s.t. $\forall x, n$.

$|p_n(x) - p_n(0)| \le c\,|x| / n^{\frac{d+1}{2}}$

Pf: By Accomp. Lemma.

Prop. (large deviations)

S_n is RW with $p \in \mathcal{P}_d$. Set $\tau_n = \min\{k \mid |S_k| \ge n\}$. $\xi_n = \min\{k \mid \mathcal{J}^*(S_k) \ge n\}$. Then:

$\exists\, t > 0.\ c < \infty$. s.t. for $\forall n$. $\forall r > 0$. s.t.

$\mathbb{P}(\tau_n \le rn^2) + \mathbb{P}(\xi_n \le rn^2) \le c\, e^{-t/r}$.

$\mathbb{P}(\tau_n \ge rn^2) + \mathbb{P}(\xi_n \ge rn^2) \le c\, e^{-rt}$.

(2) **Conti. Case:**

Def: $\bar{p}_t(x) = e^{-\mathcal{J}^*(x)^2/2t} / (2\pi t)^{\frac{d}{2}} |\Gamma|^{\frac{1}{2}}$.

Thm. (Local CLT)

If $p \in \mathcal{P}_d$. Then for $\forall k \geq 4$. $\exists c(k) < \infty$. $\forall x$.

s.t. $|\tilde{p}_t(x) - \bar{p}_t(x)| \leq c(k)\left((|z|^k + 1)\, e^{-\mathcal{J}^*(z)^2/2} + t^{-\frac{k-3}{2}}\right) / t^{\frac{d+2}{2}}$

Where $z = x/\sqrt{t}$

Pf: Consider $\tilde{p} = \tilde{p}_1$. satisfies the discrete local CLT. So for $\tilde{p}_n$.

With $\tilde{p}_{t+n}(x) = \sum_y \tilde{p}_n(x-y)\, \tilde{p}_t(y)$.

Lemma. (Strong local CLT for Poisson variable)

$N_t \sim \text{Poisson}(t)$. $m \in \mathbb{Z}^1$. s.t. $|m-t| \leq \frac{t}{2}$.

Then. $\mathbb{P}(N_t = m) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(m-t)^2}{2t}} \cdot e^{O(t^{-\frac{1}{2}} + \frac{|m-t|^3}{t^2})}$

Thm. If $\tilde{S}_t$ is conti-time one-dim SRW.

Then. for $|x| \leq \frac{t}{2}$. $\tilde{p}_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} e^{O(t^{-\frac{1}{2}} + \frac{|x|^3}{t^2})}$

Pf: Consider $x = 2k$ (Odd is similar)

Note $\tilde{p}_t(2k) = \sum_{m \geq 0} \mathbb{P}(N_t = 2m)\, \mathbb{P}(S_{2m} = 2k)$

Apply the Lemma. above.