Approxi. by BM.

Thm. (Donsker's functional CLT)

$(X_n)_{n\geq 1}$ i.i.d. with zero mean and variance 1. $S_n \triangleq \sum_1^n X_k$. Then:

$(W^{(n)}(t))_{[0,1]} =: (S_{[nt]}/\sqrt{n})_{[0,1]} \xrightarrow{d} (W_t)_{[0,1]}$

$\sim$ $SBM_{[0,1]}$

$\Rightarrow$ We can use SRW to approxi. BM.

Actually, the universe is also workable.

(1) Dimension $= 1$:

Def: i) $\tau = \inf\{t \geq 0 \mid |B_t| = 1\}$. B_t is 1-dim SBM.

ii) $\tau_0 = 0$. $\tau_n = \inf\{t \geq \tau_{n-1} \mid |B_t - B_{\tau_{n-1}}| = 1\}$.

Set $S_n =: B_{\tau_n}$. $S_t =: S_n + (t-n)(S_{n+1} - S_n)$.

Then: S_n is one-dim SRW ($\mathbb{P}(B_\tau = 1) = \frac{1}{2}$).

It's Skorokhod embedding.

Thm. There exists $0 < c, a < \infty$. s.t. $\forall r \leq n^{\frac{1}{4}}$ and $\forall n \geq 3$. $\mathbb{P}(\max_{t \leq n} |S_t - B_t| > r n^{\frac{1}{4}} \sqrt{\log n}) \leq c\, e^{-ar}$

Thm. (Conti. case)

$N_t \sim \mathrm{Poi}(1)$ indept of (B_t). Set $\tilde{S}_t = S_{N_t}$. Then $\tilde{S}_t$ distributes as conti. time SRW. and $\exists\ 0 < c, a < \infty$. s.t. for $\forall r \le n^{\frac{1}{4}}$. $\forall n \ge 0$. $\mathbb{P}(\sup_{t \le n} |\tilde{S}_t - B_t| \ge r n^{\frac{1}{4}} \sqrt{\log n}) \le c e^{-ar}$

(2) Higher dimension : $d > 1$:

Lemma. $(B^{(k)})_{1 \le k \le l}$ are indept 1-dim SBM. and $(v^k)_{1 \le k \le l} \subset \mathbb{R}^d$. Then. $\sum_1^l B_t^{(k)} \cdot v^{(k)}$ is BM in $\mathbb{R}^d$ with cov. $(v_1 \cdots v_l)\begin{pmatrix} v_1^T \\ \vdots \\ v_l^T \end{pmatrix}$

Thm. For $p \in \mathcal{P}_d$ with cov. I. Then $\exists\ c, a$. and prob. space $(\Omega, \mathcal{F}, \mathbb{P})$. where $\exists$ BM $(\vec{B}_t)$ with cov. I. SRW S_n and $\tilde{S}_t$. with dist. increment p. s.t. $\forall\ 1 \le r \le n^{\frac{1}{4}}$.

$\mathbb{P}(\max_{t \le n} |S_t - B_t| \ge r n^{\frac{1}{4}} \sqrt{\log n})$. $\mathbb{P}(\square) \le c e^{-ar}$

Pf: $B_t = \sum_1^l B_t^k \cdot v_k$. Set: $(S_n^k)_{1 \le k \le l}$ is Skorokhod embed of $(B^k)_{1 \le k \le l}$.

Let $S_n = \sum S_{L_n^k}^k \cdot v^k$.

RMK: We can't directly embed S_n into B_t. Since B_t will visit origin infinite times.

(3) Dyadic coupling:

Thm If $p \in \mathcal{P}_1'$ satisfies: $\exists b > 0$. s.t. $\mathbb{E} X_1^2 = \sigma^2$.

$\mathbb{E}(e^{b|X_1|}) < \infty$. Then. $\exists$ prob. space $(\Omega, \mathcal{F}, \mathbb{P})$.

a BM (B_t) with var. para. σ^2. and RW.

S_n. with incre. p. s.t. $\forall \alpha < \infty$. $\exists c_\alpha$. s.t.

$$\mathbb{P}(\max_{j \le n} |S_j - B_j| \geq c_\alpha \log n) \leq c_\alpha \cdot n^{-\alpha}.$$

Rmk: $\mathbb{E}(e^{b|X_1|}) < \infty$ is necessary condition:

Pf: Note: $\mathbb{P}(|B_1| \geq \hat{c} \log n) = o(n^{-1})$. by

Chebyshev. Inequi.

$$\Rightarrow \mathbb{P}(|S_1| \geq 2\hat{c} \log n) \leq \mathbb{P}(|B_1| \geq \hat{c} \log n)$$

$$+ \mathbb{P}(|S_1 - B_1| \geq \hat{c} \log n) \leq 2\hat{c}/n.$$

So: $\mathbb{P}(|X_1| \geq x) \leq 2\hat{c}\, e^{-x/2\hat{c}}$.

Thm. (High dimension)

If $p \in \mathcal{P}_d$. Then $\exists (\Omega, \mathcal{F}, \mathbb{P})$. a BM (B_t)

on $\mathbb{R}^d$ with Cov Γ. and conti. RW. $\tilde{S}_t$

with incre. p. s.t. $\forall \alpha < \infty$. $\exists c_\alpha$. We have

$$\mathbb{P}(\max_{1 \le j \le n} |\tilde{S}_j - B_j| \geq \alpha \log n) \leq c_\alpha n^{-\alpha}.$$

Pf: $B_t = \sum_1^l B^i_{q_i t} x_i$. $\tilde{S}_t = \sum_1^l S^i_{q_i t} x_i$

Decompose into SBM. and SRW. by Thm above