Green's Function

(1) Recurrence and Transience:

Def: For RW S_n with $p \in \mathcal{P}_d \cup \mathcal{P}_d^*$, it's recurrent if $\mathbb{P}(S_n = 0, \text{ i.o.}) = 1$.

Thm: i) $p \in \mathcal{P}_d'$, $d = 1, 2$, is recurrent.

ii) $p \in \mathcal{P}_d^*$, $d \geq 3$ is transient.

iii) $\mathbb{P}(S_n \neq 0, \forall n \geq 1) =: q = 1 / \sum_{n \geq 0} p_n(0)$.

Pf: i), ii) are from local CLT.

iii) Set $Y = \sum I\{S_n = 0\}$. $\mathbb{P}(Y = 0) = q$.

By Markov. $\Rightarrow Y$ is memoryless

So $Y \sim \text{Geo}(q)$.

(2) Functions:

① Green generating func.:

Def: i) If $p \in \mathcal{P} \cup \mathcal{P}^*$, $x, y \in \mathbb{Z}^d$. Green generating function $G(x, y, \xi) =: \sum_{n \geq 0} \xi^n p_n(x-y)$.

Denote $G(0, x, \xi) \triangleq G(x, \xi)$. $G(x, 1) \triangleq G(x)$.

Rmk: Interpretation for ξ: consider $T_\xi \sim \text{Geo}(1-\xi)$, i.e. killing time with parameter $1-\xi$.

$\Rightarrow G(x,\xi) = \mathbb{E}\left(\sum_{n<T} I\{S_n = x\}\right)$.

ii) For $\tilde{p}_t$. conti. time RW. Set $\tilde{G}(x,y,\xi) =: \int_0^\infty \xi^t \tilde{p}_t(x-y)\,dt$

Rmk: Similarly, we can interpret it as expectation of time spent at y. by RW with p start at x. before the killing time $\tilde{T}_\xi \sim \exp(-\log(1-\xi))$

prop. If $p \in \mathcal{P}_d^*$ is transient. Then $\int_0^\infty \tilde{p}_t(x) = G(x)$.

Pf: $\int_0^\infty I\{\tilde{S}_t = x\}\,dt = \sum I\{S_n = x\}\,(T_{n+1} - T_n)$.

Def: For $p \in \mathcal{P}_d$. lazy walk p_ε is defined by:

$$p_\varepsilon(x) = \begin{cases} (1-\varepsilon)p(x), & x \neq 0 \\ \varepsilon + (1-\varepsilon)p(0), & x = 0. \end{cases}$$

Rmk: Lazy walk will be aperiodic. Besides, $\mathcal{L}_\varepsilon = (1-\varepsilon)\mathcal{L}$. $\phi_\varepsilon(\theta) = \varepsilon + (1-\varepsilon)\phi(\theta)$. $\Gamma_\varepsilon = (1-\varepsilon)\Gamma$. $G_\varepsilon(x) = \dfrac{G(x)}{1-\varepsilon}$ if p is transient. (by prop. below).

Def: i) $\bar{\tau}_A = \min\{j \geq 0 \mid S_j \in A\}$. $\tau_A = \min\{j \geq 1 \mid S_j \in A\}$. Set $\bar{\tau}_{A/\{\eta\}} \triangleq \bar{\tau}_\eta$. $\tau_{A/\{\eta\}} \triangleq \tau_\eta$.

ii) $f_n(x,\eta) \triangleq \mathbb{P}^x(\tau_\eta = n)$

iii) First visiting function is $F(x,\eta,\zeta)$

$= \sum \zeta^n f_n(\eta - x)$.

Rmk: interpretation for ζ: T_ζ is def as before. $\Rightarrow F(x,\eta,\zeta) = \mathbb{P}^x(\tau_\eta < T_\zeta)$.

prop. i) For $n \geq 1$. $p_n(\eta) = \sum_{j=1}^{n} f_j(\eta) p_{n-j}(0)$.

ii) For $\zeta \in \mathbb{C}$. $G(\eta,\zeta) = \delta(\eta) + F(\eta,\zeta) \cdot G(0,\zeta)$.

Pf: i) By Markov prop.

ii) It's directly by i):

Note $\sum_{n\geq 1} p_n(x)\zeta^n = \sum_{n\geq 1} f_n(x)\zeta^n \sum_{m\geq 0} p_m(0)\zeta^m$

For $\eta = 0$. We have to add $p_0(0) = 1$.

Rmk: Note: $1 - F(0;\zeta) = G(0,\zeta)^{-1}$

$= \mathbb{P}(\tau_0 \geq T_\zeta) \overset{\text{if } \zeta=1}{=} q$

prop. For $p \in \mathcal{P}_d \cup \mathcal{P}_d^*$ with ch.f ϕ. Then.

for $|\zeta|<1$. $G(x,\zeta) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-ix\cdot\theta} d\theta / (1-\zeta\phi(\theta))$

if $d\geq 3$. it holds for $\zeta=1$. as well.

<u>Pf:</u> By $p_n(x) = \frac{1}{(2\pi)^d}\int \phi(\theta)^n e^{-ix\cdot\theta} d\theta$.

Check the convergence and exchange limit.

<u>Rmk:</u> We can express $G(x,\zeta)$ in terms of ch.f. $\phi(\theta)$.

<u>prop.</u> If $p\in \mathcal{P}_d \cup \mathcal{P}_d'$, $d=1,2$. $q(n) = \mathbb{P}(S_j\neq 0,\ j\leq n)$

Then $q(n) \sim \begin{cases} r\pi^{-1} n^{-\frac{1}{2}}, & d=1 \\ r(\log n)^{-1}, & d=2. \end{cases}$ $(n\to\infty)$

where $r = (2\pi)^{\frac{d}{2}}\sqrt{|\Gamma|}$.

<u>Pf:</u> It follows from Tauberian Thms.

<u>Cor.</u> For $p\in \mathcal{P}_d'$. $\bar{\tau} = \min\{n\geq 1 \mid S_n=0\}$.

$\Rightarrow \mathbb{E}(\bar{\tau}) = \infty$.

<u>Rmk:</u> It means even if $d=1,2$. RW is recurrent. it's still null recurrent.

② <u>Green Functions:</u>

Next. we investigate $G(x,q) =: G(1,x,q)$. for $p\in \mathcal{P}_d$. $d\geq 3$.

Rmk: i) $G(x,y) = G(y,x)$ is from pt $\mathbb{Z}^d$.

ii) $G(x) = \mathbb{P}(\tau_x < \infty)\, G(0)$

iii) $\mathcal{L} G(x) = -\delta(x)$.

Thm. (Asymptotic of Green Func.)

$$G(x) = \frac{C_d^*}{\mathcal{J}^*(x)^{d-2}} + O(|x|^{-d}) = \frac{C_d}{\mathcal{J}(x)^{d-2}} + O(|x|^{-d})$$

as $|x| \to \infty$. $C_d^* = d^{\frac{d}{2}-1} C_d = \dfrac{\Gamma(\frac{d}{2})}{(d-2)\pi^{\frac{d}{2}} |\Gamma|^{\frac{1}{2}}}$.

Pf: 1) $G(x) = \sum_{n \geq 1} \bar{p}_n(x) + O(|x|^{-d})$.

2) $\sum_{n \geq 1} \bar{p}_n(x) = C_d^* / \mathcal{J}^*(x)^{d-2} + O(|x|^{-d})$.

For 1), separate $\sum = \sum_{n > |x|} + \sum_{n \leq |x| \leq n^2} + \sum_{n^2 < |x|}$.

Apply Local CLT.

Rmk: By CLT: $p_n(x) \sim \frac{c}{n^{d/2}} e^{-|x|^2/2n}$.

$\Rightarrow$ position scale $R \sim$ time scale R^2.

So RW will visit $O(R^2)$ point in $\bar{B}_R$.

prob. of particular point is visited

$\sim O(R^{2-d})$.

Cor. $\nabla_j G(x) = O(|x|^{-d+1})$. $\nabla_j^2 G(x)\ O(|x|^{-d})$.

Thm. (Wanker Condition)

For $p \in \mathcal{P}_d'$.

i) $d=3$. $\Rightarrow$ $G(x) = C_d^* / \mathcal{J}^*(x)^{d-2} + O(|x|^{-1})$

Moreover. if: $\mathbb{E}|X_1|^3 < \infty$. $\Rightarrow G(x) = \square + O(\frac{\log|x|}{|x|^2})$

if $\mathbb{E}|X_1|^4 < \infty$. $\mathbb{E}X_1^3 = 0$. $\Rightarrow G(x) = \square + O(|x|^{-2})$

ii) $d \geq 3$, $\mathbb{E}(|X_1|^{d+1}) < \infty \Rightarrow G(x) = \square + O(|x|^{1-d})$

$d \geq 3$. $\mathbb{E}(|X_1|^{d+3}) < \infty \Rightarrow G(x) = \square + O(|x|^{-d})$

(3) Potential Kernels:

① Dim = 2:

Def: i) For $p \in \mathcal{P}_2^*$. $a(x,y) =: \sum_{n \geq 0} (p_n(0) - p_n(x,y))$

$= \lim_{N \to \infty} [\sum_{n \geq 0}^N p_n(0) - \sum_{n \geq 0}^N p_n(x,y)]$.

Rmk: i) By Cor. of local CLT:

$|p_n(0) - p_n(x-y)| \leq c|x-y| / n^{\frac{3}{2}}$.

$\Rightarrow$ The first term is abs. convergent.

ii) Note: $p_n(0) \sim n^{-\frac{2}{2}}$.

$\Rightarrow a(x) \neq \sum_{n \geq 0} p_n(0) - \sum_{n \geq 0} p_n(x)$.

ii) For $p \in \mathcal{P}_2$. bipartite. Set.

$$a(x) =: \begin{cases} \sum_{n \geq 0} (p_n(0) - p_n(x)) = \lim_{N \to \infty} \square & . \ x \in (\mathbb{Z}^2)_e \\ \lim_{N \to \infty} (\sum_0^N p_n(0) - \sum_0^N p_n(x)) & . \ x \in (\mathbb{Z}^2)_o \end{cases}$$

Rmk: Note for $x \in (\mathbb{Z}^2)_0$. $P_n(0) - P_n(x) \geq 0$ may not hold. (And it may not abs conv.)

$\Rightarrow$ First equation of Def in i) may not converge.

prop. For $p \in P_2'$. $\Rightarrow$ $2a(x) = \mathbb{E}^x(\sum_{n \geq 0} I\{S_n = x\})$

Pf: From an equation lately:

$G_{\mathbb{Z}^d/\{0\}}(x,x) = F_{\mathbb{Z}^d/\{0\}}(x) + a(x) = 2a(x)$.

prop. For $p \in P_2$ $\Rightarrow$ $\mathcal{L}a(x) = \delta_0(x)$.

prop. If $p \in P_2 \cup P_2^*$. Then $a(x) = (2\pi)^{-2} \int_{[-\pi,\pi]^2} \frac{1 - e^{ix\cdot\theta}}{1 - \phi(\theta)} d\theta$.

Rmk: For lazy walk p_ε def before.

We have $a_\varepsilon(x) = a(x)/(1-\varepsilon)$.

Thm. (Asymptotic of $a(x)$)

If $p \in P_2$. Then $\exists\, C = C(p)$. as $|x| \to \infty$.

$a(x) = (\log \mathcal{J}^*(x))/\pi |\Gamma|^{\frac{1}{2}} + C + O(|x|^{-2})$.

Cor. $\nabla_j a(x) = O(|x|^{-1})$. $\nabla_j^2 a(x) = O(|x|^{-2})$.

Denote: $\bar{a}(x) = (\pi |\Gamma|)^{-1} \log \mathcal{J}^*(x)$.

Thm. (Wenker conditions)

For $p \in \mathcal{P}_2'$ $\Rightarrow$ $a(x) = \bar{a}(x) + o(\log|x|)$. Moreover,

if $\mathbb{E}|X_1|^3 < \infty$ $\Rightarrow$ $\exists C < \infty$. s.t. $a(x) = \bar{a}(x) + C + O(|x|^{-1})$;

if $\mathbb{E}|X_1|^6 < \infty$. $\mathbb{E}X_1^3 = 0$ $\Rightarrow$ $\exists \tilde{C} < \infty$. s.t. $a(x) = \bar{a}(x)$
$+ \tilde{C} + O(|x|^{-2})$

② **Dim = 1:**

Def: For $p \in \mathcal{P}_1$. the potential kernel. $a(x) =:$

$$\lim_{N \to \infty} \left[\sum_{n=0}^{N} p_n(0) - \sum_{n=0}^{N} p_n(x) \right].$$

Prop. (Asymptotic estimates)

For $p \in \mathcal{P}_1'$. Then:

i) If $\mathbb{E}|X_1|^3 < \infty$. $\overset{\exists c}{\Rightarrow}$ $|a(x)\sigma^2 - |x|| \le c \log|x|$.

ii) If $\mathbb{E}|X_1|^4 < \infty$. $\mathbb{E}X_1^3 = 0$. $\overset{\exists c}{\Rightarrow}$ $a(x) = |x|/\sigma^2$
$+ C + O(|x|^{-1})$.

Rmk: For $d=1$. $a(x)$ is not as useful as $d=2$ or Green function. since it's not harmonic. So with a bad estimate.

Prop. For SRW $p \in \mathcal{P}_1$ $\Rightarrow$ $a(x) = |x|$.

(4) <u>Fundamental Solutions:</u>

For $p \in \mathcal{P}_d$. G for $d \geq 3$ and a for $d = 2$ are called fundamental solution of $\mathcal{L}$.

<u>Note:</u> $\mathcal{L}\, G(x) = -\delta(x)$ $\quad \mathcal{L}\, a(x) = \delta(x)$.

<u>Rmk:</u> i) Symmetry of RW p is necessary

ii) They're also called inverse of $\mathcal{L}$.

<u>prop.</u> If $p \in \mathcal{P}_d$. $d \geq 2$. $f : \mathbb{Z}^d \to \mathbb{R}'$. st.

$f(0) = 0$. $f(x) = o(|x|)$ $(x \to \infty)$. $\mathcal{L} f(x) = 0$.

for $x \neq 0$. Then. $\exists\, b \in \mathbb{R}'$. St.

$$\begin{cases} f(x) = b\,(G(x) - G(0)) & d \geq 3 \\ f(x) = b\, a(x). & d = 2. \end{cases}$$

<u>Pf:</u> By uniqueness of harmonic func.

<u>Rmk:</u> $f(x) = o(|x|)$ is necessary. Since $f(x_1, \cdots x_d) = x_1$ also holds.

(5) <u>Equations for Green func.</u>

<u>Denote:</u> $G_A(x, y) =: \mathbb{E}^x\left(\sum_{n=0}^{\tau_A - 1} I_{\{S_n = y\}} \right)$

Lemma. For $p \in \mathcal{P}_d$. $A \subsetneq \mathbb{Z}^d$.

i) $G_A(x,y) = G_A(y,x)$ ii) $G_A(x,y) = 0$. if x or $y \notin A$.

iii) $f(y) =: G_A(x,y)$. $\mathcal{L} f(y) = -\delta(y-x)$

iv) $\forall y \in A$. $G_A(y,y) = 1 / \mathbb{P}^y(\tau_A < \tau_y) < \infty$.

v) $G_A(x,y) = \mathbb{P}^x(\bar{\tau}_y < \tau_A) G_A(y,y)$.

vi) $G_A(x,y) = G_{A-x}(0, y-x)$.

<u>Pf:</u> iv) is from last exit formula.

<u>Prop.</u> For $p \in \mathcal{P}_d$. $A \subset \mathbb{Z}^d$. $x, y \in A$.

i) If $d \geq 3$. $\Rightarrow G(x,y) = G_A(x,y) + \mathbb{E}^x(G(S_{\bar{\tau}_A}, y), \bar{\tau}_A < \infty)$

ii) If $d = 1, 2$. A is finite. $\Rightarrow G_A(x,y) =$

$\mathbb{E}_x(a(S_{\bar{\tau}_A}, y)) - a(x,y)$.

<u>Pf:</u> By $\sum_{n \geq 0} I\{S_n = y\} = \sum_{n < \bar{\tau}_A} + \sum_{n \geq \bar{\tau}_A}$

with strong Markov property.

<u>Def.</u> For $A \subsetneq \mathbb{Z}^d$. $B_n = \{x \in \mathbb{Z}^d \mid |x| < n\}$.

$$F_A(x) =: \begin{cases} \lim_{n \to \infty} n\, \mathbb{P}^x(\tau_{B_n} < \bar{\tau}_A) / \sigma^2 & d = 1 \\ \lim_{n \to \infty} \mathbb{P}^x(\tau_{B_n} < \bar{\tau}_A) \log n / 2|\Gamma|^{\frac{1}{2}} & d = 2. \end{cases}$$

prop. If $p \in \mathcal{P}_d$, $d = 1, 2$, $A \subsetneq \mathbb{Z}^d$. Then, we have

$$G_A(x, \eta) = \mathbb{E}_x(a(S_{\bar{\tau}_A}, \eta)) - a(x, \eta) + F_A(x).$$

Pf: WLOG, set $x \in A$. Choose $n > |x| \vee |\eta|$.

Restrict on $A_n = A \cap \{|x| < n\}$.

So $G_{A_n}(x, \eta) = \mathbb{E}_x(a(S_{\bar{\tau}_{A_n}}, \eta)) - a(x, \eta)$

$\mathbb{E}_x(a(S_{\bar{\tau}_{A_n}}, \eta)) =: M_n + N_n$.

$M_n = \mathbb{E}_x(a(S_{\tau_A}, \eta), \tau_A < \tau_{B_n}) \xrightarrow{n \to \infty} \mathbb{E}_x(\square)$

$N_n = \mathbb{E}_x(a(S_{\tau_{B_n}}, \eta), \tau_A > \tau_{B_n}) \sim \mathbb{P}^x(\square) \cdot \square$

by $S_{\tau_{B_n}} \sim n$ and estimate of a.

$\Rightarrow$ Set $n \to \infty$

Rmk: i) Set $\eta \notin A$. $\Rightarrow F_A(x) = a(x, \eta) - \mathbb{E}_x(a(S_{\bar{\tau}_A}, \eta))$

So: F_A is harmonic.

ii) Set $A = \mathbb{Z}^d / \{0\}$, $\eta = 0$ $\Rightarrow F_{\mathbb{Z}^d/\{0\}}(x) = a(x)$

prop. (Last exit decomposition)

If $p \in \mathcal{P}_d$, $A \subset \mathbb{Z}^d$.

i) For $A \subset A' \subsetneq \mathbb{Z}^d$, we have:

$$\mathbb{P}^x(\bar{\tau}_{\mathbb{Z}^d/A} < \bar{\tau}_{A'}) = \sum_{z \in A} G_{A'}(x, z)\, \mathbb{P}^z(\tau_{\mathbb{Z}^d/A} > \tau_{A'})$$

ii) For $\xi \in (0, 1)$, $T_\xi \sim \mathrm{Geo}(1-\xi)$, indept. r.v.

Then, $\mathbb{P}^x(\bar{\tau}_{\mathbb{Z}^d/A} < T_\xi) = \sum_{z \in A} G(x, z, \xi)\, \mathbb{P}^z(\tau_{\mathbb{Z}^d/A} > T_\xi)$.

Pf: Only prove i). ii) is similar.

Set $\sigma = \max\{k < \tau_{A'} \mid S_k \in A\}$. Assume $x \in A'$.

$$LHS = \sum_{k\geq 0} \sum_{z\in A} \mathbb{P}^x(\sigma = k,\ S_\sigma = z)$$

$$= \sum_{z\in A} \sum_{k\geq 0} \mathbb{P}^x(S_k = z,\ k < \tau_{A'},\ S_j \notin A,\ j: k+1 \sim \tau_{A'})$$

$$\mathbb{P}^x(S_j \notin A,\ j: k+1 \sim \tau_{A'} \mid S_k = z,\ k < \tau_{A'})$$

$$= \mathbb{P}^z(\tau_{A'} < \tau_{\mathbb{Z}^d/A}) \quad \text{By Markov.}$$

Cor. For $d \geq 3$, $|A| < \infty$, we have:

$$\mathbb{P}^x(\tau_{\mathbb{Z}^d/A} < \infty) = \sum_{z\in A} G(x,z)\, \mathbb{P}^z(\tau_{\mathbb{Z}^d/A} = \infty)$$

Pf: Set $\zeta \to 0$ in ii)

prop. For $p \in \mathcal{P}_d$, $0 \in A \subset \mathbb{Z}^d$, $\zeta \in (0,1)$, $T_\zeta \sim \mathrm{Geo}(1-\zeta)$, indept of S_n. Set $\ell = \max\{0 \leq j \leq \tau_A \mid S_j = 0\}$ and $\ell^* = \max\{0 \leq j \leq T_\zeta\}$. Then,

i) $\{S_j \mid \ell \leq j \leq \tau_A\} \overset{d}{\sim} \{S_j \mid 0 \leq j \leq \tau_A\} \mid \ell = 0$

ii) $\{S_j \mid \ell^* \leq j \leq T_\zeta\} \overset{d}{\sim} \{S_j \mid 0 \leq j \leq T_\zeta\} \mid \ell^* = 0$.

Pf: i) For $x_1 \sim x_{k-1} \in A/\{0\}$, $x_k \in \mathbb{Z}^d/A$.

$$\mathbb{P}(\ell = j,\ \tau_A = j+k,\ S_{j+1} \sim S_{j+k} = x_1 \sim x_k)$$

$$= \mathbb{P}(\ell = j,\ \tau_A > j,\ \square) = \mathbb{P}(\ell = j,\ \tau_A > j)\, \mathbb{P}(\square \mid \ldots)$$

$$\overset{mp}{=} \mathbb{P}(\ell = j,\ \tau_A > j)\, \mathbb{P}(S_1 = x_1, \ldots S_k = x_k \mid \ell = 0)$$

Sum up j: $\sum \mathbb{P}(\ell = j,\ \tau_A > j) = \mathbb{P}(\tau_A > \ell) \overset{0\in A}{=} 1$