Potential Theory

(1) Definitions:

Def: For $A \subsetneq \mathbb{Z}^d$. $p \in P$.

i) $\partial A =: \{x \in \mathbb{Z}^d / A \mid p(\eta, x) > 0$. for some $\eta \in A\}$.

is outer boundary of A. $\bar{A} =: A \cup \partial A$.

ii) $\partial^i A =: \partial(\mathbb{Z}^d / A)$ is inner boundary of A.

iii) A is connected if $\forall x, \eta \in A$. $\exists p_n(x, \eta) > 0$.

iv) $f : \bar{A} \to \mathbb{R}^1$ is p-harmonic in A if:

$$\mathcal{L} f(\eta) = \sum_x p(\eta, x)(f(x) - f(\eta)) = 0. \ \forall \eta \in A.$$

prop. If f is harmonic in $\mathbb{Z}^d$. w.r.t. p.

increment of S_n. Then:

$$M_n =: f(S_n) - \sum_{j=0}^{n-1} \mathcal{L} f(S_j) \text{ is mart.}$$

Cor. If f is harmonic in A. Then.

$$f(S_{n \wedge \tau_A}) \text{ is mart.}$$

prop. $p \in P_d$. $f : \mathbb{Z}^d \to \mathbb{R}^1$ is bdd and harmonic

in $\mathbb{Z}^d$. $\Rightarrow f$ is const.

Pf: Set $\hat{p} = \frac{1}{2}\delta_0 + \frac{1}{2}p$. aperiodic.

$\Rightarrow$ f is also $\hat{p}$-harmonic.

Set S. $\hat{S}$. are RWs. start at x. y.

Note that: $|f(x) - f(y)| =$

$|\mathbb{E}(f(S_n)) - \mathbb{E}(f(\hat{S}_n))| \leq 2c\|f\| \cdot |x-y| / n^{\frac{1}{2}}$

$\to 0$. as $n \to \infty$

Rmk: It's closely related to that RW will forgets its starting point.

(2) Dirichlet Problem:

Thm. (First Version)

If $p \in \mathcal{P}_d$. $A \subset \mathbb{Z}^d$. satisfies $P^x(\tau_A < \infty) = 1$. for $\forall x \in A$. $F: \partial A \to \mathbb{R}^1$ is bdd. Then. $\exists$ unique bdd func. $f: \bar{A} \to \mathbb{R}^1$. $f(x) = \mathbb{E}^x(F(S_{\tau_A}))$ st. $\mathcal{L}f(x) = 0$ in A. $f(x) = F(x)$ on ∂A.

Pf. Note $f(S_{n \wedge \tau_A})$ is bdd mart.

Apply the optional Sampling Thm

Rmk: i) If A is finite. Then $\bar{A}$ is finite.

$\Rightarrow$ All func's on $\bar{A}$ will be finite.

So: There's unique func. satisfies these conditions.

ii) If A is infinite. There may be more than one solutions.

e.g. $A = \mathbb{Z}^{>0}$. $F(0) = 0$. Then. $(f_b(x))_{b\in\mathbb{R}'} =: (bx)_{b\in\mathbb{R}'}$ is set of solutions.

iii) When $d=1,2$. $A \subsetneq \mathbb{Z}^d$. By recurrent. A satisfies the condition

iv) When $d \geq 3$. $\mathbb{Z}^d / A$ is finite. Then. $f(x) =: \mathbb{P}^x(\bar{\tau}_A = \infty)$ is bdd function satisfies the conditions with $F=0$. on ∂A. $\Rightarrow$ The solution isn't unique.

v) It's analogue of conti. version. When $f(x) = \mathbb{E}^x(f(B_T))$

Thm. (Second Version)

If $p \in \mathcal{P}_d$. $A \subsetneq \mathbb{Z}^d$. $F: \partial A \to \mathbb{R}'$ is bdd. Then. the only bdd func. $f: \bar{A} \to \mathbb{R}'$. Satisfies the Dirichlet conditions are of form:

$f(x) = \mathbb{E}^x(F(S_{\bar{\tau}_A}); \bar{\tau}_A < \infty) + b\,\mathbb{P}^x(\bar{\tau}_A = \infty)$ for some $b \in \mathbb{R}'$.

Rmk: As Rmk iv) above. b can be chosen arbitrarily in some case.

<u>Lemma</u>: If $p \in \mathcal{P}_d$. $A \subset \mathbb{Z}^d$. $\mathbb{P}^x(\bar{\tau}_A = \infty) > 0$. for some $x \in A$. Then. $\forall \varepsilon > 0$. $\exists y \in A$. s.t.

$\mathbb{P}^y(\bar{\tau}_A = \infty) > 1 - \varepsilon$.

<u>Pf</u>: 1) A is infinite.

When $d \le 2$. the condition won't hold. unless $A = \mathbb{Z}^d$.

When $d \geq 3$. by transience $\Rightarrow |A| = \infty$

2) Set $\widetilde{B}_k = A \cap (B_{2^k} / B_{2^{k-1}})$.

At least infinite $\widetilde{B}_k \neq \emptyset$.

If $\exists q > 0$. s.t. $\forall y \in A$. $\mathbb{P}^y(\bar{\tau}_A = \infty) \le 1 - q$.

i.e. $\inf_y \mathbb{P}^y(\bar{\tau}_A < \infty) \geq q > 0$

$\mathbb{P}^x(\bar{\tau}_A < \infty) = \mathbb{P}^x(\exists k.\ \bar{\tau}_A < \infty.$ when S_n walks in $\widetilde{B}_k)$

$\geq \mathbb{P}^x(\bar{\tau}_A < \infty.$ in $\widetilde{B}_k$. i.o.$)$

But. $\sum_k \mathbb{P}^x(\bar{\tau}_A < \infty$ in $B_k) \geq \sum_{\widetilde{B}_k \neq \emptyset} q = \infty$.

$\Rightarrow \mathbb{P}^x(\bar{\tau}_A = \infty) = 0$. contradiction!

<u>Return to pf</u>:

WLOG. Set p is aperiodic. $\mathbb{P}^x(\bar{\tau}_A = \infty) > 0$.

Assume f is bdd satisfying the Dirichlet condition.

1°) Note $f(x) = \mathbb{E}_x(f(S_{n\wedge\tau_A}))$

$= \mathbb{E}_x(f(S_n)) - \mathbb{E}_x(f(S_n), \tau_A < n) + \mathbb{E}_x(\square)$.

$\Rightarrow |f(x)-f(y)| \le 2\|f\|_\infty (\mathbb{P}^x(\tau_A<\infty) + \mathbb{P}^y(\tau_A<\infty))$

2°) Set $U_\varepsilon = \{z\in\mathbb{Z}^d \mid \mathbb{P}^z(\tau_A=\infty)\ge 1-\varepsilon\} \ne \emptyset$. $\forall \varepsilon>0$.

By 1°). $|f(x)-f(y)| \le 4\varepsilon\|f\|_\infty$. $\forall x,y\in U_\varepsilon$

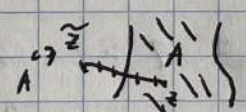
$\Rightarrow \exists b$. s.t. $|f(x)-b|\le 4\varepsilon\|f\|_\infty$. $\forall x\in U_\varepsilon$

3°) Set $\ell_\varepsilon = \tau_A \wedge \inf\{n\ge 0 \mid S_n\in U_\varepsilon\} \triangleq \tau_A\wedge\tilde\ell_\varepsilon$.

By optional sampling Thm: $f(x) = \mathbb{E}_x(f(S_{\ell_\varepsilon}))$

$\triangleq A + B = \mathbb{E}(F(S_{\tau_A}). \tau_A \le \ell_\varepsilon) + \mathbb{E}(f(S_{\ell_\varepsilon}) \cdots)$

Note $U_\varepsilon \downarrow \emptyset$ as $\varepsilon\to 0$. $\Rightarrow \tilde\ell_\varepsilon\to\infty$. $\varepsilon\to 0$.

($\mathbb{P}^z(\tau_A<\infty) \overset{\forall z}{>} 0$.  . $\exists$ path $z\to\tilde z$)

$A\to \mathbb{E}_x(F(S_{\tau_A}). \tau_A<\infty)$. as $\varepsilon\to 0$.

$|B - b\,\mathbb{P}^x(\tau_A>\ell_\varepsilon)| \overset{2°)}{\le} 4\varepsilon\|f\|_\infty \mathbb{P}^x(\tau_A>\ell_\varepsilon) \xrightarrow{\varepsilon\to 0} 0$

Rmk: It's generalization of first version.

b can be written as $F(\infty)$.

Def: For $p\in\mathcal{P}_d$. $A\subset\mathbb{Z}^d$. Poisson kernel is H: $\bar A\times\partial A \to [0,1]$. def by $H_A(x,y) = \mathbb{P}^x(\tau_A<\infty.\ S_{\tau_A}=y)$. Denote $H_A(x,\infty) = \mathbb{P}^x(\tau_A=\infty)$

Rmk: i) Note that $\sum_{y\in\partial A} H_A(x,y) = \mathbb{P}^x(\tau_A<\infty)$.

ii) $f(x) =: H_A(x,y)$ is func. on $\bar{A}$ is harmonic. If p is recurrent $\Rightarrow f$ is unique. If p is transient $\Rightarrow f$ is unique func. s.t. $\to 0$. as $x \to \infty$.

Pf: $H_A(x,y) \leq \mathbb{P}^x(\tau_y < \infty)$

$$= G(x,y)/G(x,x) \asymp \frac{1}{|x-y|^{d-2}} \to 0.$$

iii) $\mathbb{E}^x(F(S_{\tau_A})) = \sum_{y \in \partial A} H_A(x,y)\, F(y)$

prop. If $p \in \mathcal{P}_d$. $A \subsetneq \mathbb{Z}^d$. $g: A \to \mathbb{R}^1$ is func. with finite support. Then $f(x) = \mathbb{E}_x\left(\sum_{0 \le j < \tau_A} g(S_j)\right)$ is unique bdd func. on $\bar{A}$. s.t.

$f(x) = 0$ on ∂A. $\mathcal{L} f(x) = -g(x)$ on A.

Pf: $M_n =: f(S_{n \wedge \tau_A}) + \sum_{j=0}^{\tau_A \wedge n-1} g(S_j)$ is mart.

cor. If $p \in \mathcal{P}_d$. $A \subsetneq \mathbb{Z}^d$. finite set. $F: \partial A \to \mathbb{R}^1$. $g: A \to \mathbb{R}^1$. are given. Then $f(x) = \mathbb{E}_x(F(S_{\tau_A}))$ $+ \mathbb{E}^x\left(\sum_0^{\tau_A - 1} g(S_j)\right)$ is unique bdd func. on $\bar{A}$

s.t. $\mathcal{L} f(x) = -g(x)$ on A. $f(x) = F(x)$ on ∂A.

(3) Properties of harmonic:

Denote: $B_n = \{|z|<n\}$. $C_n = \{\mathcal{J}(x) < n\}$.

$\xi_n =: \tau_{C_n}$. $\xi_n^* =: \tau_{B_n}$.

prop. For $p\in \mathcal{P}_d$, $d\geq 3$, $x\in \partial C_n \cup \partial^i C_n$. Then

$G(x) = \frac{C_d}{n^{d-2}} + O(n^{1-d})$, $d\geq 3$.

$a(x) = C_2 \log n + \gamma_2 + O(n^{-1})$, $d=2$

where C_d, C_2, γ_2 are defined as before

Pf: $\mathcal{J}(x) = n + O(1)$ for $x\in \partial C_n \cup \partial^i C_n$.

$\Rightarrow$ $(n+O(1))^{d-2} = n^{d-2} + O(n^{1-d})$.

$\log(n+O(1)) = \log n + O(n^{-1})$.

prop. For $p\in\mathcal{P}_d$, $G_{C_n}(0,0) = G(0,0) - \frac{C_d}{n^{d-2}} + O(n^{1-d})$, $d\geq 3$

and $G_{C_n}(0,0) = C_2\log n + \gamma_2 + O(n^{-1})$, $d=2$, holds

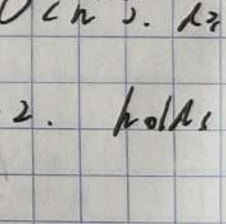
Pf: By $G_{C_n}(0,x) = G(0,x) + \mathbb{E}(G(S_{\tau_{C_n}}, x))$, $d\geq 3$

$G_{C_n}(0,x) = \mathbb{E}(a(S_{\tau_{C_n}}, x))$, $d=2$

Lemma. For $p\in\mathcal{P}_d$, $d\geq 2$. Then $\forall \varepsilon>0$, $r<\infty$, $\exists c$ s.t. if $B_{\varepsilon n}\subset A \subsetneq \mathbb{Z}^d$, then $\forall |x| > n\varepsilon$, $|y|\leq r$.

$|G_A(0,x) - G_A(y,x)| \leq c/n^{d-1}$

$|2G_A(0,x) - G_A(y,x) - G_A(-y,x)| \leq c/n^d$.



Pf: By the equations used in the last prop.

Lemma. For $p \in \mathcal{P}_d$, $d \geq 2$. Then $\exists\, C_1, C_2 < \infty$.

s.t. for n large enough, we have:

$\mathbb{P}^x(S_{\tau_{C_n / C_{\frac{n}{2}}}} \in C_{n/2}) \geq C_1/n$. if $x \in \bar{C}_n / C_{n/2}$

$\mathbb{P}^x(S_{\tau_{C_n / C_{n/2}}} \in C_{n/2}) \leq C_2/n$. if $x \in \partial C_n$.

Rmk: The exit prob. can be bounded

depending on how close x near

the boundary of C_n.

Prop. (last-exit formula)

If $p \in \mathcal{P}_d$. $x \in B \subset A \subsetneq \mathbb{Z}^d$. $\eta \in \partial A$. Then.

$H_A(x,\eta) = \sum_{z \in B} G_A(x,z)\, \mathbb{P}^z(S_{\tau_{A/B}} = \eta)$.

$H_A(x,\eta) = \sum_{z \in A} G_A(x,z)\, \mathbb{P}(z,\eta)$. in particular.

Pf: $\mathbb{P}^x(\tau_A < \infty,\ S_{\tau_A} = \eta) = \mathbb{P}^x(\tau_B < \tau_A < \infty,\ S_{\tau_{A/B}} = \eta)$

$= \sum_{k, z \in B} \mathbb{P}(\tau_B = k,\ \tau_A > k,\ S_{\tau_B} = z,\ S_{\tau_{A/B}} = \eta)$

$\overset{M.P.}{=} \sum_{z \in B} \mathbb{P}^z(S_{\tau_{A/B}} = \eta) \sum_{k \geq 0} \mathbb{P}(\tau_A > k,\ S_k = z)$

The second is similar. $B = A / \partial^i A$.

Lemma. If $p \in \mathcal{P}_d$. Then for n large enough.

$H_{C_n}(x,\eta) \asymp n^{1-d}$ for $x \in C_{n/4}$, $\eta \in \partial C_n$.

Pf: By last-exit formula.

Thm. (Difference estimate)

If $p \in \mathcal{P}_d$. $r < \infty$. Then. $\exists c$. s.t. for n large enough. The following holds:

i) $g : \bar{B}_n \to \mathbb{R}^1$. harmonic in B_n. $|\eta| \le r$.

$|\nabla_\eta g(0)| \le c \|g\|_\infty / n$. $|\nabla_\eta^2 g(0)| \le c\|g\|_\infty / n^2$.

ii) $f : \bar{B}_n \to \mathbb{R}^{\geq 0}$. harmonic in B_n. $|\eta| \le r$.

$|\nabla_\eta f(0)| \le c f(0)/n$. $|\nabla_\eta^2 f(0)| \le c f(0)/n^2$.

Pf: Only prove i). ii) is similar.

By MVT. for $C_{2\varepsilon n} \subset B_n$. $B_r \subset C_{\frac{r\varepsilon}{2}}$.

$g(x) = \sum_{z \in \partial C_{2n\varepsilon}} H_{C_{2\varepsilon n}}(x, z)\, g(z)$.

Set $h(x) = H_{C_{2\varepsilon n}}(x, z)$. $\Rightarrow$ prove it for h.

By last exit formlar on $C_{2\varepsilon n} \supset C_{\varepsilon n}$.

with estimate of H (so: $f(x) \le c f(\eta)$)

Thm. (Harnack inequility)

If $p \in \mathcal{P}_d$. $U \subset \mathbb{R}^d$. open connected. $K \subset U$. cpt. Then. there exists $c = c(K, U, p) < \infty$. and $N = N(K, U, p)$. for $\forall n \geq N$. and $f : \overline{nU} \to \mathbb{R}_{\geq 0}$. harmonic in nU. we have: $f(x) \le c f(\eta)$. for $\forall x, \eta \in nK$.

Pf: By estimate above. $f(x) \le c_0 f(y)$.

if $|x-y| \le \delta \cdot d(x, \partial U_n)$. for some c_0.

Def: z, w are adjacent if $|z-w|$

$\le \frac{\delta}{4} \cdot d(z, \partial U_n) \vee d(w, \partial U_n)$.

Set $e(z,w) = \min\{k \ge 0 \mid \exists (z_i)_1^k.\ z_1 = z,$

$z_k = w$. s.t. (z_i, z_{i+1}) is adjacent$\}$.

Set $V_k = \{w \in U \mid e(z,w) \le k\} \Rightarrow U = \bigcup_{k \ge 0} V_k$.

By cpt of K. cover it by finite V_k.

prop. If $p \in P_d$. $d \ge 3$. $\mathbb{Z}^d / A$ is finite. $f: \mathbb{Z}^d \to \mathbb{R}^1$

is harmonic and $f(x) = o(|x|)$ as $x \to \infty$.

Then. $\exists b \in \mathbb{R}^1$ for $\forall x$. $f(x) = \mathbb{E}_x(f(S_{\bar{\tau}_A}). \bar{\tau}_A < \infty)$

$+ b\, \mathbb{P}^x(\bar{\tau}_A = \infty)$

Lemma. If $p \in P_d$. $d \ge 3$. $m \le n/4$. $C_n / C_m \subset A \subset C_n$.

$x \in C_{2m}$. $\mathbb{P}^x(S_{\tau_A} \in \partial C_n) > 0$. $z \in \partial C_n$. Then.

$\mathbb{P}^x(S_{\tau_A} = z \mid S_{\tau_A} \in \partial C_n) = H_{C_n}(0, z)(1 + O(\frac{m}{n}))$

for n large enough.

Pf: WLOG. Set $0 \in A$. $f(x) \equiv 0$ on $\mathbb{Z}^d / A$.

(Or set $\bar{f}(x) = f(x) - \mathbb{E}_x(f(S_{\bar{\tau}_A}). \bar{\tau}_A < \infty)$)

(*) Note $0 = f(0) = E(f(S_{\xi_n})) - \sum_{\mathbb{Z}^d / A} h_{C_n}(0, y) \Delta f(y)$.

Set $n \to \infty$. Then $\lim_n \mathbb{E}(f(S_{\xi_n})) =: b$

$= \sum_{\eta \in \mathbb{Z}^2/A} h(0,\eta) \, f(\eta)$. (by $\mathbb{Z}^2/A$ is finite)

2°) By optional sampling:

$f(x) = \mathbb{P}^x(\tau_A > \xi_n) \, \mathbb{E}^x(f(S_{\xi_n}) \mid \tau_A > \xi_n)$

for $x \in A \cap C_n$.

3°) By sublinearity of f: $\exists \, \varepsilon_n \xrightarrow{n\to\infty} 0$. s.t.

$|f(x) - f(y)| \leq 2n\varepsilon_n$. for $x, y \in \partial C_n$.

4°) $\mathbb{E}^x(f(S_{\xi_n}) \mid \tau_A > \xi_n) - \mathbb{E}(f(S_{\xi_n})) \overset{w \in \partial C_n}{=}$

$\sum_{z \in \partial C_n} (f(z) - f(w))(\mathbb{P}^x(S_{\xi_n} = z \mid \tau_A > \xi_n) - H_{C_n}(0,z))$

Sum up $w \in \partial C_n$. and divided $|\partial C_n|$.

Note by Lemma. $|\mathbb{P}^x(S_{\xi_n} = z \mid \tau_A > \xi_n) - H_{C_n}(0,z)|$

$\leq c \cdot \frac{|x|}{n} H_{C_n}(0,z)$.

$\Rightarrow |\mathbb{E}^x(f(S_{\xi_n}) \mid \tau_A > \xi_n) - \mathbb{E}(f(S_{\xi_n}))|$

$\leq c \cdot \frac{|x|}{n} \sup_{\partial C_n} |f(x) - f(y)| \leq c|x|\varepsilon_n \to 0$.

5°) By 4°) Set $n \to \infty$. in 2°).

<u>prop.</u> If $p \in \mathcal{P}_2$. $A \subsetneq \mathbb{Z}^2$. finite. contain origin. Then,

$g_A(x) =: \lim_{n\to\infty} C_2 \log n \cdot \mathbb{P}^x(\xi_n < \bar{\tau}_{\mathbb{Z}^2/A})$ exists.

and $g_A(x-y) = a(x-y) - \mathbb{E}^x(a(S_{\bar{\tau}_{\mathbb{Z}^2/A}} - y))$ for $y \in A$.

Rmk: We can express the limit of exit prob. into potential kernel.

prop. If $p \in \mathcal{P}_2$, $A \subset \mathbb{Z}^2$ finite. $f: \mathbb{Z}^2 \to \mathbb{R}^1$ is harmonic on $\mathbb{Z}^2/A$, vanishes on A, and $f(x) = o(|x|)$ $(x \to \infty)$. Then, $f = b\, g_A$ for some b.

Pf: 1) $\mathbb{E}(f(S_{\xi_n})) = \sum_{y \in A} h_{C_n}(0, y)\, \mathcal{L}f(y)$

$= c_2 \log n \sum_A \mathcal{L}f(y) + O(1)$.

2) As 4) step in former prop.

$|\mathbb{E}(f(S_{\bar{\tau}_{\mathbb{Z}^2/A} \wedge \xi_n}) \mid \xi_n < T) - \mathbb{E}(f(S_{\xi_n}))|$

$\leq c \frac{|x| \log n}{n} \sup_{\partial C_n} |f(z) - f(y)| \leq c|x| \varepsilon_n \log n$.

3) By prop. above. $f(x) = \mathbb{E}^x(f(S_{\bar{\tau}_{\mathbb{Z}^2/A} \wedge \xi_n}))$

$= g_A(x) \sum_A \mathcal{L}f(y) + O(1)$.

(4) Capacity:

① Dimension ≥ 3:

Def: Let $T_A = \tau_{\mathbb{Z}^d/A}$, $\bar{T}_A = \bar{\tau}_{\mathbb{Z}^d/A}$ for $A \subset \mathbb{Z}^d$.

If $p \in \mathcal{P}_d$, $d \geq 3$. $e_A(x) =: \mathbb{P}^x(T_A = \infty)$ and

$g_A(x) =: \mathbb{P}^x(\bar{T}_A = \infty)$

Rmk: i) $e_A(x) = 0$ if $x \in A / \partial^i A$

ii) $q_A(x)$ is unique func on $\mathbb{Z}^d$ that is zero on A. harmonic on $\mathbb{Z}^d / A$. St. $q_A(x) \to 1$ as $x \to \infty$.

iii) $\mathcal{L} q_A(x) = e_A(x)$. by Markov prop.

Define capacity of A is $\mathrm{cap}(A) = \sum_{x \in A} e_A(x)$

prop. (Motivation of Def)

For $p \in \mathcal{P}_d$. $d \geq 3$. $A \subset \mathbb{Z}^d$. finite. Then: $\mathbb{P}^x(T_A < \infty) = \frac{C_d \, \mathrm{cap}(A)}{\mathcal{J}(x)^{d-2}} \cdot (1 + O(\frac{\mathrm{rad}(A)}{|x|}))$ for $|x| \geq 2 \, \mathrm{rad} A$

Lemma. $\mathbb{P}^x(\overline{T}_A < \infty) = \sum_{y \in A} G(x, y) \, e_A(y)$.

Pf: By last exit decomposition

prop. For $p \in \mathcal{P}_d$. $d \geq 3$. $\mathrm{cap}(C_n) = n^{d-2} / C_d + O(n^{d-1})$

Pf. Note $1 = \mathbb{P}(\overline{T}_{C_n} < \infty) = \sum_{C_n} G(0, y) \, e_{C_n}(y)$

with $G(0, y) = C_d \, n^{2-d} (1 + O(n^{-1}))$

prop. If $p \in \mathcal{P}_d$. $d \geq 3$. $A, B \underset{\text{finite}}{\subset} \mathbb{Z}^d$. Then: we have,

$\mathrm{cap}(A \cup B) \leq \mathrm{cap}(A) + \mathrm{cap}(B) - \mathrm{cap}(A \cap B)$

Pf: $\mathbb{P}^x(T_{A \cup B} < \zeta_n) = \mathbb{P}^x(T_A < \zeta_n$, or $T_B < \zeta_n)$

$$= \mathbb{P}^x(T_A < \zeta_n) + \mathbb{P}^x(T_B < \zeta_n) - \mathbb{P}^x(T_A < \zeta_n, T_B < \zeta_n)$$

$$\le \mathbb{P}^x(T_A < \zeta_n) + \mathbb{P}^x(T_B < \zeta_n) - \mathbb{P}^x(T_{A\cap B} < \zeta_n)$$

<u>Lemma.</u> $\mathrm{cap}(A) = \lim_{n\to\infty} \sum_{y\in\partial C_n} \mathbb{P}^y(T_A < \zeta_n)$

<u>Pf:</u> By symmetry of path.

$$\sum_{x\in A} \mathbb{P}^x(T_A > \zeta_n) = \sum_A \sum_{\partial C_n} \mathbb{P}^x(S_{T_A \wedge \zeta_n} = y)$$

$$= \sum_{\partial C_n} \mathbb{P}^y(\zeta_n > T_A)$$

So set $n\to\infty$, obtain the result.

<u>Def:</u> For $p\in \mathcal{P}_d$, $d\geq 3$, $A\subset \mathbb{Z}^d$, the harmonic measure of A is $\mathrm{hm}_A(x) = e_A(x)/\mathrm{cap}(A)$, $x\in A$.

<u>prop.</u> If $A\subset C_{n/2}$, $y\notin C_n$. Then we have:

$$\mathbb{P}^y(S_{T_A} = x \mid T_A < \infty) = \mathrm{hm}_A(x)\left(1 + O\left(\frac{\mathrm{rad}(A)}{|y|}\right)\right)$$

So: $\lim_{y\to\infty} \mathbb{P}^y(S_{T_A} = x \mid T_A < \infty) = \mathrm{hm}_A(x)$.

<u>prop.</u> For $p\in \mathcal{P}_d$, $d\geq 3$, $A\subset \mathbb{Z}^d$ finite. Then:

$$\mathrm{cap}(A) = \sup\{\sum_A f(x) \mid f\geq 0,\ \mathrm{supp}(f) = A,\ Gf\leq 1\}.$$

$$= \inf\{\sum_A f(x) \mid f\geq 0,\ \mathrm{supp}(f) = A,\ Gf\geq 1\}.$$

<u>Pf:</u> By Lemma above, proved as before.

Prop. For $p, q \in \mathcal{P}_d$, $d \geq 3$. cap_p, cap_q are corresponding capacity. Then $\exists\, \delta = \delta(p,q)$, s.t. $\forall A \underset{\text{finite}}{\subset} \mathbb{Z}^d$, $\delta\, \mathrm{cap}_p(A) \leq \mathrm{cap}_q(A) \leq \delta^{-1} \mathrm{cap}_p(A)$.

Rmk: capacity of different RW are comparable in same dimension.

Def: For $p \in \mathcal{P}_d$, $d \geq 3$. $A \subset \mathbb{Z}^d$ is recurrent if $\mathbb{P}(S_n \in A, \text{ i.o.}) = 1$.

Rmk: i) Finite sets are transient.

ii) Finite union of transient sets is transient.

iii) Note $\{S_n \in A, \text{ i.o}\}$ is exchangable $\overset{\text{0-1 law}}{\Longrightarrow} \mathbb{P}(S_n \in A, \text{ i.o}) \in \{0,1\}$.

iv) $\sum_{x \in A} G(x) < \infty \Rightarrow A$ is transient.

Pf: $LHS = \mathbb{E}(\sum I_{\{S_n \in A\}})$

Lemma. For $p \in \mathcal{P}_d$, $d \geq 3$, $A \subset \mathbb{Z}^d$. Then A is transient $\Leftrightarrow \sum_{k \geq 1} \mathbb{P}(T^k < \infty) < \infty$, where $T^k =: T_{A_k}$, $A_k = A \cap (C_{2^k} / C_{2^{k-1}})$.

Pf: Set $E_k = \{T_k < \infty\}$. Note:

A is transient $\Leftrightarrow$ $\mathbb{P}(E_k \text{ i.o}) = 0$.

Since RW is transient. So it will visit infinite (A_k).

So we obtain $(\Leftarrow)$ by BC-Lemma.

Cor. (Wiener's test.)

If $p \in \mathcal{P}_d$. $d \geq 3$. $A \subset \mathbb{Z}^d$. Then A is transient $\Leftrightarrow$ $\sum_{k \geq 1} 2^{(2-d)k} \operatorname{cap}(A_k) < \infty$.

Rmk: If A is transient w.r.t some $p \in \mathcal{P}_d$. Then A is transient w.r.t all $p \in \mathcal{P}_d$.

Pf: $\mathbb{P}(T^k < \infty) \asymp 2^{(2-d)k} \operatorname{cap}(A_k)$.

Thm. If $d \geq 3$. $p \in \mathcal{P}_d$. S_n is p-walk. $A =: S[0, \infty)$.

Then A is recurrent if $d = 3, 4$. it's transient if $d \geq 5$.

Rmk: Note that in B_n. the number of points visited by RW $\asymp n^2$.

$\Rightarrow$ trajectory of RW can be seen as 2-dim set.

It's equi. to ask if 2 2-dim sets will intersect.

KOKUYO

② Dimension = 2:

Def: Use the prop. before to define $g_A(x)$ in $p \in \mathcal{P}_2$. $g_A(x) =: a(x) - \mathbb{E}_x(a(S_{\overline{T}_A}))$

prop. For $p \in \mathcal{P}_2$. $\lim_{|\eta| \to \infty} \mathbb{P}^\eta(S_{T_A} = x) = \mathcal{L} g_A(x)$.

Def: i) For $p \in \mathcal{P}_2$. harmonic measure of A is $hm_A(x) =: \lim_{|\eta| \to \infty} \mathbb{P}^\eta(S_{T_A} = x)$.

Rmk: Note $\mathbb{P}^n(T_A < \infty) = 1$. in dim = 2. it's consistent with def before.

ii) capacity of A is $\mathrm{cap}(A) =: \lim_{\eta \to \infty} (a(\eta) - g_A(\eta)) = \lim_{\eta \to \infty} \mathbb{E}_\eta(a(S_{\overline{T}_A})) = \sum_{x \in A} hm_A(x) a(x)$.

Rmk: By estimate of $a(x) \sim \log|x| + \gamma_2$:

$\lim_{\eta \to \infty} a(\eta - z) - a(\eta) = 0$. So for $z \in A$.

$\mathrm{cap}(A) = \sum_{x \in A} hm_A(x) a(x - z)$.

prop. For $p \in \mathcal{P}_2$

i) If $A \subset B \subset \mathbb{Z}^d$. Then. $g_A(x) \geq g_B(x)$. $\forall x$. and $\mathrm{cap}(A) \leq \mathrm{cap}(B)$.

ii) If $A \cdot B \subset \mathbb{Z}^d$. Then. $g_{A \cup B}(x) \geq g_A(x) + g_B(x) - g_{A \cap B}(x)$. and $\mathrm{cap}(A \cup B) \leq \mathrm{cap}(A) + \mathrm{cap}(B) - \mathrm{cap}(A \cap B)$

prop. For $p \in P_2$. $A \subset \mathbb{Z}^2$. Then:

$$\mathrm{cap}(A) = 1/\sup\{\sum_A f(y) \mid f \geq 0.\ a_A f \leq 1 \text{ on } A\}$$

$$= 1/\inf\{\sum_A f(y) \mid f \geq 0.\ a_A f \geq 1 \text{ on } A\}.$$

where $a_A f(x) =: \sum_{y \in A} a(x, y) f(y)$.

prop. For $p \in P_2$. $\mathrm{cap}(C_n) = C_2 \log n + \gamma_2 + O(n^{-1})$.

Pf: Use asymptotic estimate of g_{C_n}.

prop. For $p \in P_2$. $A \subset B \underset{\text{finite}}{\subset} \mathbb{Z}^2$. Then. $\mathrm{cap}(A)$

$= \mathrm{cap}(B) - \sum_B hm_B(y) g_A(y)$.

Pf: Note $g_A - g_B$ is bdd. harmonic in $\mathbb{Z}^2/B$.

with boundary value g_A on B. $(*)$

$$\mathrm{cap}(A) - \mathrm{cap}(B) = \lim_{x \to \infty} (g_B(x) - g_A(x))$$

$$= \lim_{x \to \infty} (E_x(g_B(S_{\bar{T}_B}) - g_A(S_{\bar{T}_B})))$$

$$\overset{(*)}{=} \sum_B hm_B(y) g_A(y)$$

(5) Neumann Problem:

Def: i) For $p \in P_d$. $A \subset \mathbb{Z}^d$. $f: \bar{A} \to \mathbb{R}'$. its normal derivative at $y \in \partial A$ is:

$$Df(y) = \sum_{x \in A} p(y, x)(f(x) - f(y))$$

ii) Given $D^*: \partial A \to \mathbb{R}^1$. the Neumann problem is to find $f: \bar{A} \to \mathbb{R}^1$. st.

$$\begin{cases} \mathcal{L} f = 0 & \text{on } A. \\ Df = D^* & \text{on } \partial A. \end{cases}$$

<u>Lemma.</u> If $p \in \mathcal{P}_d$. $A \underset{\text{finite}}{\subset} \mathbb{Z}^d$. $f: \bar{A} \to \mathbb{R}^1$. Then:

$$\sum_{x \in A} \mathcal{L} f(x) = - \sum_{\eta \in \partial A} Df(\eta)$$

<u>Rmk:</u> So a necessary of existence of solution is $\sum_{\partial A} D^*(\eta) = 0$. it's a LS with dimension $= \#\partial A - 1$.

<u>Pf:</u> $LHS = \sum_{x \in A} \sum_{\eta \in \bar{A}} p(x, \eta) (f(\eta) - f(x))$

$$= \sum_{x \in A} \sum_{\eta \in A} + \sum_{x \in A} \sum_{\eta \in \partial A}$$

But $\sum_{x, \eta \in A} = 0$. by the sym of RW.

<u>Def:</u> $A \subset \mathbb{Z}^d$. excursion Poisson kernel is $H_{\partial A}: \partial A \times \partial A \to [0,1]$. define by $H_{\partial A}(\eta, z) = \mathbb{P}^\eta(S_1 \in A, S_{\tau_A} = z) = \sum_{x \in A} p(\eta, x) H_A(x, z)$.

<u>Lemma.</u> For $f: \bar{A} \to \mathbb{R}^1$ harmonic. in A. Then:

$$Df(\eta) = \sum_{z \in \partial A} H_{\partial A}(\eta, z) (f(z) - f(\eta)).$$

Pf: RHS $= \sum_{z\in\partial A} \sum_{x\in A} p(\eta,x) H_A(x,z)(f(z)-f(\eta))$

$= \sum_{x\in A} p(\eta,x) \left(\sum_{z\in\partial A} H_A(x,z) f(z) - f(\eta) \right)$

$=$ LHS.

Cor. For $H(x) = H_A(x,z)$. Then:

$DH(\eta) = H_{\partial A}(\eta, z)$. for $\eta \in \partial A / \{z\}$.

$DH(z) = H_{\partial A}(z,z) - \mathbb{P}^z(S_1 \in A)$.

Rmk: Note $\sum_{z\in\partial A} H_{\partial A}(\eta, z) = \mathbb{P}^\eta(S_1 \in A) \leq 1$

Set $\widehat{H}_{\partial A}(\eta,z) = H_{\partial A}(\eta,z)$. $\eta \neq z$.

and $\widehat{H}_{\partial A}(z,z) = 1 - \sum_{\eta\in\partial A, \eta\neq z} \widehat{H}_{\partial A}(\eta,z)$

$\Rightarrow Df(\eta) = \sum_{z\in\partial A} \widehat{H}_{\partial A}(\eta,z)(f(z)-f(\eta))$

still holds. i.e. $Df = (\widehat{H}_{\partial A} - I) f$

Lemma. Rank of $\widehat{H}_{\partial A} - I$ is $\#\partial A - 1$.

Pf: Note $\forall c \in \mathbb{R}$. $(\widehat{H}_{\partial A} - I)(f+c) = (\widehat{H}_{\partial A} - I) f$

$\Rightarrow$ dim ker $(\widehat{H}_{\partial A} - I) \geq 1$.

But when $\widehat{H}_{\partial A} - I$ restricts on the space of harmonic func. its images will contain a LS with dim $= \#\partial A - 1$.

So: dim ker $(\widehat{H}_{\partial A} - I) = 1$.

Def: For $p \in \mathcal{P}_d$. define a random walk "reflected off ∂A" by: $q(x,y) = p(x,y)$. $x, y \in A$. $q(x,y) = p(x,y)$. if $x \neq y \in \partial A$. $q(x,x) = 1 - \sum_{\substack{z \in \bar{A} \\ z \neq x}} q(x,z)$.

Rmk: The RW can move as p in A.

When in $y \in \partial A$. it only moves to $A \cup \{y\}$

prop. For $p \in \mathcal{P}_d$. $A \subset \mathbb{Z}^d$. finite. connected. $D^*: \partial A \to \mathbb{R}^1$. satisfies $\sum_{\partial A} D^*(y) = 0$. Then $\exists f$ solves Neumann problem for D^*. which's unique up to a const. one such func. is:

$f(x) = - \lim_{n \to \infty} \mathbb{E}^x (\sum_{j=0}^{n} D^*(Y_j) I_{\{Y_j \in \partial A\}})$. where $Y \sim q$ as def above.

Pf: 1) Uniqueness is from Lemma. above

2) q is irred. sym. having stat. dist.

$\pi(x) = 1 / \# \partial A$.

It's easy to check $f(x)$ satisfies the condition after proving it's convergent.

$\mathbb{E}^x (\sum_{j=0}^{n} D^*(Y_j) I(Y_j \in \partial A)) = \sum_{j=0}^{n} \sum_{z \in \partial A} (q_j(x,z) - \frac{1}{m}) D^*(z)$

with $|q_j(x,z) - \pi(z)| \overset{\text{Doeblin}}{\leq} c e^{-\alpha j}$