LERW and Loops

(1) Loop measure:

① **Def:** Given trans. weight P. m, m^*. rt. unrt measure.

i) $\mathcal{L}_j$ is set of rooted loop with length j.

$\mathcal{L}_j(A) =: \{ w \in \mathcal{L}_j \mid w \subset A \}$. $\mathcal{L}_j^x(A) =: \mathcal{L}_j(A) \cap \{ x \in w \}$

$\mathcal{L} =: \bigcup_j \mathcal{L}_j$. $\mathcal{L}(A) = \bigcup_j \mathcal{L}_j(A)$. $\mathcal{L}^x(A) =: \bigcup_j \mathcal{L}_j^x(A)$

Denote $\mathcal{L}^*$ is notation for unrooted.

ii) For tree $\mathcal{T}$ with root x_0. $p(\mathcal{T}; x_0) =: \prod_{(x,x') \in \mathcal{T}} p(x, x')$

Rmk: If P is sym. Then $p(\mathcal{T}; x_0)$ is indept of choice of root.

Def: (generating func.)

i) $\tilde{g}(\lambda, x) =: \sum_{\substack{w \in \mathcal{L} \\ w_0 = x}} \lambda^{|w|} p(w) =: \sum p_\lambda(w)$

Rmk: λ can be seen as killing weight.

ii) $l(w) =: \# \{ j \mid w_0 = w_j, j \geq 1 \}$

$f(\lambda, w) =: \sum_{\substack{w \in \mathcal{L} \\ w_0 = x \\ l(w) = 1}} \lambda^{|w|} p(w)$. first return g.f.

Rmk: Easy to check $\tilde{g}(\lambda, x) = 1 + f(\lambda, x)\tilde{g}(\lambda, x)$

iii) $\tilde{g}(\lambda) =: \sum_{w \in \mathcal{L}} |\lambda|^{w} p(w)$

iv) $\phi(\lambda) =: \sum_{w \in \mathcal{L}} \lambda^{|w|} m(w) = \sum_{\mathcal{L}} \frac{\lambda^{|w|}}{|w|} p(w)$.

loop measure g.f.

v) For A is finite set. $V \subset A$.

$F(A, \lambda) =: \exp\left(\sum_{\substack{w \in \mathcal{L}(A) \\ |w| \geq 1}} p(w) \lambda^{|w|} / |w| \right)$

$F_V(A, \lambda) =: \exp\left(\sum_{\substack{w \in \mathcal{L}(A) \\ w \cap V \neq \emptyset}} p(w) \lambda^{|w|} / |w| \right)$

prop. For $V = \{\eta_1, \cdots \eta_k\} \subset A$. Then:

$F_V(A, \lambda) = F_{\eta_1}(A, \lambda) \cdot F_{\eta_2}(A / \{\eta_1\}, \lambda) \cdots F_{\eta_k}(A / \{\eta_i\}_1^{k-1}, \lambda)$

Pf: As same when we decompose G_D.

Prop. $F_x(A, \lambda) = \sum_{\substack{w \in \mathcal{L}(A) \\ w_0 = x}} \lambda^{|w|} p(w) \overset{\Delta}{=} \tilde{g}_A(\lambda, x)$.

Rmk: $\tilde{g}_A(1, x) = \mathbb{E}_x\left(\sum_0^{\tau_A - 1} I_{\{S_n = x\}} \right) = G_A(x, x)$

$\Rightarrow F_x(A, 1)^{-1} = 1 / G_A(x, x) = \mathbb{P}(\tau_x > \tau_A)$

Pf: Note that $m^*(w^*) = \sum_{w \sim w^*} \frac{p(w)}{|w|}$

$= \sum_{\substack{w \sim w^* \\ w_0 = x}} p(w) / d_x(w)$

$$\sum_{w^* \in \mathcal{I}_x(A)} m^*(w^*) \lambda^{|w^*|} = \sum_{j \geq 1} \frac{1}{j} \sum_{\substack{w \in \mathcal{L}(A) \\ w_0 = x,\ l_x(w) = j}} p(w) \lambda^{|w|}$$

$$= \sum_{j \geq 1} \frac{1}{j} \left(f_A(\lambda, x) \right)^j$$

$$= -\log(1 - f_A(\lambda, x)) = \log \tilde{g}_A(\lambda, x)$$

prop. For $|A| < \infty$. $F(A, \lambda) < \infty$. Then $F(A, \lambda) =$
$1 / |I - \lambda P|_{A \times A}|$.

Pf: WLOG. set $\lambda = 1$. prove by induction on $|A|$.

when $|A| = 1$. $F(A, 1) = \sum p^n = \tilde{g}_A(1, A)$.

For $|A| = n > 1$. $x \in A$. then:

$$\tilde{g}(1, x) = \left(\sum_{j \geq 0} P^j \right)_{x,x} = [(I - P)^{-1}]_{x,x}$$

$$= |I - P|_{x \times x}| / |I - P|_{A \times A}|$$

Use Lemma above and inductive hyp. on $A / \{x\}$.

prop. $\lambda_{0,x}$ is radius of convergence of $\tilde{g}_A(\lambda, x)$.

If p is irred. Then:

i) $\lambda_{0,x}$ is indept of x. and write λ_0.

ii) $|A| < \infty \Rightarrow \lambda_0^{-1}$ is largest eigenvalue

for $P|_{A \times A}$ (So for sub-Markovian

trans. prob. p: $\lambda_0 > 1$)

Besides. $\tilde{g}_A(\lambda_0, x) = F_x(A, \lambda_0) = \infty$.

Prop. For λ_0 the convergence of radius of $\tilde{g}$.

Then for $x \in A$. $\log F(A/\{x\}, \lambda_0) = \lim_{\lambda \to \lambda_0^-}$

$(\log F(A, \lambda) - \log \tilde{g}_A(\lambda, x))$

Rmk: Note that $F(A/\{x\}, \lambda_0) < \infty$.

when p is irred. $|A| < \infty$.

Pf: $\tilde{g}_A(\lambda, x) = F_x(A, \lambda)$. Decompose $F(A, \lambda)$.

Prop. For A is finite. p is irred. trans. prob.

with stat. dist. π which is reversible.

If $\alpha_1 = 1$. $\alpha_2 \cdots \alpha_n$ is eigenvalues of $P|_{A \times A}$

Then. $\forall x \in A$. $1/F(A/\{x\}) = \pi(x) \prod_{2}^{n} (1 - \alpha_j)$.

Pf: Note $\lim_{\lambda \to 1^-} \frac{\det(I - \lambda P|_{A \times A})}{1 - \lambda} = \prod_{2}^{n} (1 - \alpha_j)$

And $g_A(\lambda, x) = F_x(A, \lambda) \sim \pi(x)/(1-\lambda)$ $(\lambda \to 1^-)$

Since $g_A(\lambda, x)$ is expected number of

visiting x start at x. before killing

with rate $1 - \lambda$. And the expected step

RW exists is $1/(1-\lambda) \xrightarrow{\lambda \to 1^-} \infty$.

So $g_A(\lambda, x) \sim \pi(x) \cdot (1-\lambda)^{-1}$ as $\lambda \to 1^-$.

③ Boundary Excursion :

prop. (Measure on self-avoiding paths)

For $\eta = (\eta_0, \cdots \eta_k)$ is self-avoiding path in A. Then :

$$\hat{P}_A(\eta) =: \sum_{\substack{w \in \mathcal{L}(A) \\ LE(w) = \eta}} p(w) = p(\eta) F_\eta(A)$$

Pf: Set $A_j = A / \{\eta_0 \cdots \eta_j\}$. For $w \in \mathcal{L}(A)$. st. $LE(w) = \eta$. we have :

$$w = w^0 \oplus (\eta_0, \eta_1) \oplus w^1 \cdots \oplus (\eta_{k-1}, \eta_k) \oplus w^k.$$

where w^i is loop start at η^i in A_{i-1}.

$$\Rightarrow LHS = \sum_{w^0} p(w^0) \sum_{w^1} p(w^1) \cdots \sum p(w^k) \cdot p(\eta)$$

$$= F_{\eta_0}(A) \cdots F_{\eta_k}(A_{k-1}) \cdot p(\eta)$$

$$= F_\eta(A) \cdot p(\eta).$$

Cor. $F_{V_1, V_2}(A) =: \exp(\sum_{\substack{w \in L(A) \\ w \cap V_1 \neq \emptyset, w \cap V_2 \neq \emptyset}} p(w) / |w|)$ for $V_1, V_2 \subset A$.

If $A_1 \subset A$. $\eta = (\eta_0, \cdots \eta_k)$ is self-avoiding path in A_1. Then :

$$\hat{P}_A(\eta) = \hat{P}_{A_1}(\eta) F_{\eta, A/A_1}(A).$$

Pf: Note $F_\eta(A) = F_\eta(A_1) F_{\eta, A/A_1}(A)$

Def: In graph $\mathcal{X}$, given weight p.

i) $\partial A =: (\partial A)_p = \{ \eta \in \mathcal{X} / A \mid p(x,\eta) + p(\eta,x) > 0,\ \exists x \in A \}$.

boundary excursion in A is a path $w = (w_0, \cdots w_k)$, $k \geq 2$, s.t. $w_0, w_k \in \partial A$, $w_i \in A$, $i \neq 0, k$.

ii) Denote $\mathcal{E}_A(x,\eta) = \{ w \mid w_0 = x,\ w_k = \eta \}$.

$$\mathcal{E}_A =: \bigcup_{x,\eta \in \partial A} \mathcal{E}_A(x,\eta).$$

$\hat{\mathcal{E}}$ is definition for self-avoiding path.

iii) $p|_{\mathcal{E}_A}$ is excursion measure on A.

$p|_{\hat{\mathcal{E}}_A}$ is self-avoiding excursion measure on A.

$\hat{p}_A(\eta) =: p\{ w \in \mathcal{E}_A \mid LE(w) = \eta \}$. loop-erased excursion measure on A.

<u>Rmk</u>: i) The first two measures have restriction property.

ii) $\hat{p}_A(\eta) = F_\eta(A) \cdot p(\eta)$

Def: (general boundary Poisson kernel)

$H_{\partial A} : \partial A \times \partial A \longrightarrow [0,1]$ is given by:

$$H_{\partial A}(x,\eta) =: \sum_{w \in \mathcal{E}_A(x,\eta)} p(w) = \sum_{w \in \hat{\mathcal{E}}_A(x,\eta)} \hat{p}_A(w).$$

Consider $\Sigma_A(\vec{X},\vec{\eta}) := \prod_1^k \Sigma_A(X_i,\eta_i)$. $\vec{X}=(X_1\cdots X_k)$. $\vec{\eta}=(\eta_1,\cdots\eta_k)$

$p\times\cdots\times p$ is measure on it. for $[W]=(W^1\cdots W^k)$ $\in\Sigma_A(\vec{X},\vec{\eta})$. $p\times\cdots p([W]) = \prod_1^k p(w^i)$.

We want ~~to~~ define a noninteresecting loop-erased measure $\hat{p}_A(\vec{X},\vec{\eta})$ on it.

1) $p_A(\vec{x},\vec{\eta})([W]) =: p\times\cdots p([W])$. nonintersecting

excursion measure. $[W]\in\Sigma_A(\vec{x},\vec{\eta})$. $w^i\cap w^j=\emptyset$. $i\neq j$.

$p_A(\vec{x},\vec{\eta})|_{\hat{\Sigma}_A(\vec{x},\vec{\eta})}$ is nonintersecting self-avoiding case.

2) $\hat{p}_A(\vec{x},\vec{\eta})(\eta^1\cdots\eta^k) =: \vec{p}\{(W^1\cdots W^k)\in\Sigma_A(\vec{x},\vec{\eta}) \mid LE(w^i) = \eta^i.\ 1\le i\le k.\ W^{j+1}\cap(\eta^1\cup\cdots\eta^j)=\emptyset.\ j\le k-1\}$

<u>prop.</u> $\hat{p}_A(\vec{x},\vec{\eta})(\eta^1\cdots\eta^k) = \prod_1^k \hat{p}_A(\eta^i)\cdot I\{\eta^i\cap\eta^j=\emptyset.\ \eta^k\in\Sigma_A(x_k,\eta_k).\ \forall k.\ i<j\}\ /\ F_{\eta^1\cdots\eta^k}(A)$.

where $F_{\eta^1\cdots\eta^k}(A) = \exp\left(\sum_{w\in\mathcal{L}(A)}\frac{p(w)}{|w|}J(w;\eta^1\cdots\eta^k)\right)$

$J(w,[\eta]) = \max\{0,S-1\}$. S is number of $(\eta^i)_1^k$ intersect with w.

<u>Rmk:</u> $\hat{p}_A(\vec{x},\vec{\eta})[\eta]$ doesn't depend on the order of $[\eta]=(\eta^1\cdots\eta^k)$.

<u>pf:</u> By Def: $LHS = \prod_1^k \left(p(\eta^i)\exp\left(\sum_{\substack{w\in\mathcal{L}(A\setminus\eta^1\cup\cdots\eta^{i-1})\\|w|\ge1.\ w\cap\eta^i\neq\emptyset}} p(w)/|w|\right)\right)$

Replace to $\hat{p}_A$ by Rmk ii).

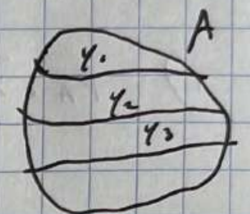
<u>prop.</u> (Fomin's Id)

If $\hat{H}_{\partial A}(\vec{x},\vec{\eta})$ is total mass of $\hat{P}_A(\vec{x},\vec{\eta})$.

Then $\sum_{z\in S_k} (-1)^{\operatorname{sgn} z} \hat{H}_{\partial A}(z(\vec{x}),\vec{\eta}) = \det(H_{\partial A}(x_i,\eta_i))_{k\times k}$

<u>Rmk:</u> If A is simply connected in $\mathbb{Z}^2$.

p is SRW. Then at most one term in LHS can be nonzero. $\overset{\text{permu}}{\Rightarrow}$ $\hat{H}_{\partial A}(\vec{x},\vec{\eta}) = \det H_{\partial A}(x_i,\eta_i)_{k\times k}$.



<u>Pf:</u> $RHS = \sum_{z} (-1)^{\operatorname{sgn} z} \prod_1^k H_{\partial A}(x_{z(i)},\eta_i)$

The id. is intuitive.

(2) <u>LERW:</u>

For $p\in P_d$. $(S_n^i)_i$ is seq of indept RW. with increment p. Set τ_A^j, $\bar{\tau}_A^j$ are corrpn exit time for S_n^j.

① <u>h-processes:</u>

<u>Denote:</u> $p_n^A(x,\eta) = \mathbb{P}^x(S_n=\eta,\ n<\bar{\tau}_A)$.

<u>Def:</u> $h:\mathbb{Z}^d\to\mathbb{R}^{\geq 0}$ is harmonic and positive on A. Vanishes on $\mathbb{Z}^d/\bar{A}$. $(\partial A)_+ = \{\eta\in\partial A \mid h(\eta)>0\}$

i) The h-process is MC on $\bar{A}$ with the trans. prob. $\widetilde{p}^{A,h}$. defined by:

$x \in \bar{A}$ $|x-\eta|=1$. $\widetilde{p}^{\square}(x,\eta) = p(x,\eta)h(\eta) / \sum_{z \sim x} p(x,z)h(z)$

Rmk: For $x \in A$. By harmonic of h

$$\Rightarrow \widetilde{p}^{\square}(x,\eta) = \frac{h(\eta)}{h(x)} \cdot p(x,\eta)$$

ii) The h-process stopped at $(\partial A)^+$ is the chain with $p^{A,h}$ equal $\widetilde{p}^{A,h}$ except $p^{A,h}(x,x)=1$. for $x \in (\partial A)^+$.

prop. $p_n^{A,h}(x,\eta) = p_n^{A}(x,\eta) \frac{h(\eta)}{h(x)}$ for $x,\eta \in A$.

Pf: Directly by Rmk above.

Rmk: h-process can be seen as RW weighted with func. $h(\cdot)$.

② Def: For $A \subset \mathbb{Z}^d$. $V \subset \partial A$. $h_{V,A}(x) =: \mathbb{P}^x(S_{\bar{\tau}_A} \in V)$

LERW from x to V in A is p.m. on the path: $LE\{h_{V,A}$-process stopped at $V\}$.

prop. For $x \in \bar{A}/V$. $(\hat{S}_0 \cdots \hat{S}_\ell)$ is LERW from x to V in A. $\eta = (\eta_0 \cdots \eta_n)$ is self-avoiding path with $\eta_0 = x \in A$. $\eta_n \in V$. $\eta_i \in A$. $i \neq 0, n$.

Then: $\mathbb{P}(\ell = n. (\hat{S}_0 \cdots \hat{S}_\ell) = \eta) = \frac{F_\eta(A) p(\eta)}{\mathbb{P}^x(S_{\tau_A} \in V)}$

Pf: By recovering the unerased path.

$\Rightarrow$ LHS $\propto$ $P(\eta) \cdot F_\eta(A)$.

With normalization $\sum_n \sum_\eta$ LHS $= \mathbb{P}^x(S_{\tau_A} \in V)$

cor. (Reversibility)

$x, y \in \partial A$. $(\hat{S}_0, \cdots \hat{S}_\ell)$ is LERW from x to y in A. Then the distribution of $(\hat{S}_\ell, \hat{S}_{\ell-1}, \cdots \hat{S}_0)$ is LERW as well

Rmk: $LE(w^R) \neq (LE(w))^R$. where R is reversal operator.

Pf: Note that $F_\eta(A)$ is indept of the order of η.

cor. For $x \in A$. $h_{V,A}(x) > 0$. Then the dist. of LERW from x to V in A stopped at V is same as LERW from x to V in $A/\{x\}$. Stopped at V.

Pf: $\sigma =: \max\{n < \tau_A \mid S_n = x\}$.

Then $(X_\sigma \cdots X_{\tau_A}) \sim h_{V,A}$ - process stopped at V. $\mid \tau_x = \infty$. $\sim h_{V,A/\{x\}} \cdots$

$(X_\sigma \cdots X_{\tau_A})$ and $(X_0, \cdots X_{\tau_A})$ generate same LERW.

Rmk: For $x \in \partial A / V$. Then the first step $\hat{S}_1 \sim$ The first step of $h_{V,A}$ process. : $\mathbb{P}^x(\hat{S}_1 = \eta) = p(x,\eta)\, h_{V,A}(\eta) / \Sigma$ ⊟

prop. $x \in \bar{A} / V$. $(\hat{S}_0, \cdots \hat{S}_\ell)$ is LERW from x to V in A. $\eta = (\eta_0 \cdots \eta_m)$ is self-avoiding path. $\eta_0 = x$. $\eta_i \in A$, $i \neq 0$. Then :

$$\mathbb{P}(\ell > m.\ (\hat{S}_0 \cdots \hat{S}_m) = \eta) = \frac{p(\eta)\, \mathbb{P}^{\eta_m}(S_{\tau_{A/\eta}} \in V)}{\mathbb{P}^x(S_{\tau_A} \in V)} F_\eta(A)$$

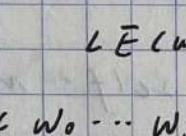
Pf: set $s = \max\{ j \mid w_j = \eta_m \}$.

where w satisfies $LE(w) = (\eta, \cdots \hat{S}_\ell)$.

$w = w^- \oplus w^+ = (w_0 \cdots w_s) \oplus (w_s \cdots w_n)$

$\Rightarrow$ Weight of w^- is : $F_\eta(A) \cdot p(\eta)$

weight of w^+ is : $\mathbb{P}^{\eta_m}(S_{\tau_{A/V}} \in V)$

Rmk. LERW is not Markov process. it's even not simply Markovian. Since the past can effect the present : e.g.  : $\hat{S}_4$ can be back to $\hat{S}_0 = 0$.

But it satisfies domain Markov (Cor. above)

③ LERW in $\mathbb{Z}^d$:

i) dimension ≥ 3 :

Def: LERW in Z^d is loop erased of RW.

Rmk: When $d \geq 3$, we can obtain infinite path by transience.

prop. For LERW in Z^d, $d \geq 3$.

i) (Domain Markov)

$\tilde{S}_{m+1} \mid (\hat{S}_0 \cdots \hat{S}_m) \sim h_{\infty, A_m}$ - process start at $\hat{S}_m$, where $A_m = Z^d / \{\hat{S}_0, \cdots \hat{S}_m\}$.

$$\mathbb{P}(\hat{S}_{m+1} = x \mid \hat{S}_0 \cdots \hat{S}_m) = \frac{p(\hat{S}_m, x)\, h_{\infty, A_m}(x)}{\sum_{y \sim x} p(x, y)\, h_{\infty, A_m}(y)}$$

ii) η is self-avoiding path, $\eta_0 = 0$.

Then: $\mathbb{P}((\hat{S}_0, \cdots \hat{S}_m) = \eta) = e_{A_m}(\eta_m) F_\eta(Z^d) p(\eta)$

Pf: **Lemma.** If Z^d / A is finite, $A^r = A \cap \{|z| < r\}$, $V^r = \partial A^r \cap \{|z| \geq r\}$, $(\hat{S}_0^{(r)}, \cdots \hat{S}_m^{(r)})$ is first m steps of LERW from 0 to V^r in A^r. Then for self-avoiding path η: $\mathbb{P}((\hat{S}_0 \cdots \hat{S}_m) = \eta) = \lim_{r \to \infty} \mathbb{P}((\hat{S}_0^{(r)} \cdots \hat{S}_m^{(r)}) = \eta)$.

Consider LERW from 0 to ∂A^r in A^r.
Apply the last prop. in ②

ii) **Dimension = 2**

Def: $\Theta_N = \{\eta = (0, \eta_1 \cdots \eta_k) \mid \eta$ is self-avoiding in $B_N\}$. $(\tilde{S}_{0,N}, \cdots \tilde{S}_{\sigma_N,N})$ is LERW from 0 to ∂B_N in B_N. ν_N is the p.m on Θ_N.

prop. For $d=2$. $n \le N$. $\eta = (0, \cdots \eta_k) \in \Theta_n$. Then:

$\nu_N(\eta) = p(\eta) \, \mathbb{P}^{\eta_k}(\bar{\zeta}_n < \zeta_{\partial}^{\eta}/\eta) \, F_\eta(B_N)$

Pf. By last prop. in ③.

prop. For $d=2$. $n<\infty$. $N \ge n$. ν_N is p.m. on Θ_n. and $\nu = \lim_{N\to\infty} \nu_N$ exists. Besides, $\forall \eta \in \Theta_n$.

$\nu_N(\eta) = \nu(\eta)(1 + O(1/\log(N/n)))$. $N \ge 2n$.

prop. For $d \ge 3$. $n<\infty$. ν_N is p.m. on Θ_n. $n \le N$. and $\nu = \lim_{N\to\infty} \nu_N$ exists. Besides. $\forall \eta \in \Theta_n$.

$\nu_N(\eta) = \nu(\eta)(1 + O((n/N)^{d-2}))$. $N \ge 2n$.

Rmk: It's speed of convergence of Lemma above.