Some Specific RWs.

(1) One-dimension RW:

① Gambler's ruin estimate:

$S_n = S_0 + \sum_1^n X_k$ in $\mathbb{Z}^1$. $X_k \overset{iid}{\sim}$, $\mathbb{E}X_1 = 0$. $\mathbb{E}X_1^2 > 0$.

Set: $\eta_r \overset{\Delta}{=} \min\{n \geq 0 \mid S_n \leq 0 \text{ or } S_n \geq r\}$. $\eta \overset{\Delta}{=} \min\{S_n \leq 0\}$.

Prop. For $j < k$. positive interger: $\mathbb{P}^j(S_{\eta_k} = k) = \frac{j}{k}$

Pf: By optional sampling thm.

Prop. $\mathbb{P}^1(\eta > 2n) \overset{i)}{=} \mathbb{P}^1(S_{2n} > 0) - \mathbb{P}^1(S_{2n} < 0)$

$\overset{ii)}{=} \mathbb{P}^1(S_{2n} = 0) = 1/\sqrt{2n} + O(n^{-\frac{3}{2}})$.

Pf: By reflection prin:

$\mathbb{P}^1(\eta \leq 2n, S_{2n} = x) = \mathbb{P}^1(\eta \leq 2n, S_{2n} = -x)$

$LHS = \sum_{x>0} \mathbb{P}^1(\eta > 2n, S_{2n} = x)$

$= \sum_{x>0} \mathbb{P}^1(S_{2n} = x) - \mathbb{P}^1(\eta \leq 2n, S_{2n} = -x)$

$\overset{i)}{=} \sum_{x>0} (p_{2n}(1, x) - p_{2n}(1, -x))$

$= p_{2n}(1, 1) + \sum_{x>0} (p_{2n}(1, x+2) - p_{2n}(1, -x))$

$\overset{ii)}{=} \mathbb{P}^1(S_{2n} = 0) = 4^{-n}\binom{2n}{n}$

prop. $\forall \varepsilon > 0$. $k < \infty$. $\exists\, 0 < c_1 < c_2 < \infty$. s.t. if

$\mathbb{P}(|X_1| \geq k) = 0$. $\mathbb{P}(X_1 \geq \varepsilon) \geq \varepsilon$. then:

$$c_1 \frac{x+1}{r} \leq \mathbb{P}^x(S_{\eta_r} \geq r) \leq c_2 \frac{x+1}{r}. \quad \forall\, 0 < x < r.$$

Rmk: c_1, c_2 may depend on k, ε.

Pf: 1°) Check $S_{n \wedge \eta_r}$ is a u.i. mart.

2°) Show $\frac{x}{r+k} \leq \mathbb{P}^x(S_{\eta_r} \geq r) \leq \frac{x+k}{r}$.

for $\forall\, k \leq x \leq r$. by 1°). ($\mathbb{E}^x(S_{\eta_r}) = x$)

3°) $\mathbb{P}^x(S_{\eta_r} \geq k) \geq \mathbb{P}^x(X_i \geq \varepsilon.\ \forall\, i \leq \lfloor \frac{k}{\varepsilon} \rfloor)$

$\geq \varepsilon^{\lfloor k/\varepsilon \rfloor}$

By markov: $\varepsilon^m \mathbb{P}^k(S_{\eta_r} \geq r) \leq \mathbb{P}^x(S_{\eta_r} \geq r) \leq \mathbb{P}^k(0)$

prop. $\forall \varepsilon > 0$. $k < \infty$ $\exists\, 0 < c_1 < c_2 < \infty$. s.t. if

$\mathbb{P}(|X_1| \geq k) = 0$. $\mathbb{P}(X_1 \geq \varepsilon) \geq \varepsilon$. then:

$$c_1 \frac{x+1}{r} \leq \mathbb{P}^x(\eta \geq r^2) \leq c_2 \frac{x+1}{r}. \quad \forall x > 0.\ r > 1.$$

③ killed walk:

Def: i) $p_\infty = 1 - \sum_k p_k$ the killing rate. $T =: \min\{j \mid S_j = \infty\}$

$\sim \mathrm{Geo}(p_\infty)$. is killing time for RW.

ii) $V_+ =: \{S_j > 0 \mid 1 \leq j \leq T-1\}$. $\bar{V}_+ =: \{S_j \geq 0 \mid 1 \leq j \leq T-1\}$.

$V_- =: \{S_j < 0 \mid 1 \leq j \leq T-1\}$. $\bar{V}_- =: \{S_j \leq 0 \mid 1 \leq j \leq T-1\}$

Rmk: $\mathbb{P}(V_+ = V_-)$, $\mathbb{P}(V_+ \cap \bar{V}_-) = \mathbb{P}(T=1) = P_\infty$.

prop. $\bar{V}_+$ is indept of V_-.

Cor. $\mathbb{P}(V_-) = \mathbb{P}(V_+) = (1-P_{0,+})\,\mathbb{P}(\bar{V}_+)$.

$= (P_\infty(1-P_{0,+}))^{\frac{1}{2}}$

where $P_{k,+} = \sum_{j\geq 1} \mathbb{P}(S_j = k,\ j<T,\ S_l<0,\ 1\leq l<j-1)$

Pf: Note $\mathbb{P}(\bar{V}_-) = \mathbb{P}(V_-) + P_{0,+}\,\mathbb{P}(\bar{V}_-)$

with $P_\infty = \mathbb{P}(V_+)\,\mathbb{P}(\bar{V}_-) = \dfrac{\mathbb{P}(V_+)^2}{1-P_{0,+}}$

Pf: $(\Rightarrow)$ prove: $\bar{V}_+^c$ is indept of V_-.

Note $\bar{V}_+^c \cap V_- = \{T>1,\ S_i \notin \{0,1,\cdots\},\ i\leq T-1\}$

Consider $l = \max\{k \mid S_j = k,\ j\leq T-1\}$.

$g_k = \max\{j \mid S_j = k,\ j\leq T-1\}$.

Use the symmetry of RW and translate $-k$ steps (let the farthest point be origin)

(2) Simple RWs:

① Poisson kernel:

Let $\mathcal{H} = \{(x,y) \in \mathbb{Z}^{d-1}\times\mathbb{Z}^1 \mid y>0\}$. $\overline{(x,y)} \triangleq (x,-y)$

prop. (For half plane)

For SRW in $\mathbb{Z}^d$. $d \geq 2$. $z, w \in \mathcal{H}$.

i) $G_{\mathcal{H}}(z, w) = G(z, w) - G(z - \bar{w})$.

$H_{\mathcal{H}}(z, 0) = \frac{1}{2d}(G(z - e_d) - G(z - \bar{e}_d))$

ii) $H_{\mathcal{H}}(z, 0) = h_{BM}(z, 0)(1 + O(\frac{|\eta|}{|z|^2})) + O(|z|^{-d-1})$.

where $h_{BM}(z, 0) = 2\eta / 2\pi^{d/2}/\Gamma(\frac{d}{2}) \cdot |z|^d$. for

$z = (x, \eta) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{>0}$. poisson kernel for BM.

③ Approxi. conti. harmonic func.

prop. $\exists c < \infty$. s.t. $\forall n, m \in \mathbb{Z}^{>0}$. $\mathcal{U} =: \{x \in \mathbb{R}^d \mid |x| < 1\}$.

$f: (n+m)\mathcal{U} \to \mathbb{R}^1$ is harmonic. Then $\exists \hat{f}$ on

$\bar{B}_n$. s.t. $\mathcal{L}\hat{f} = 0$ on B_n. and $|f(x) - \hat{f}(x)|$

$\leq c\|f\|_\infty / m^2$. $\forall x \in B_n$. $\hat{f}(x) = \mathbb{E}^x(S_{\zeta_n})$.

Pf: use mart. $S_{\zeta_n \wedge k} - \sum_{k \geq 0}^{\zeta_n - 1} \mathcal{L}f(S_k)$

prop. (Converse)

(f_n) seq. func. on $\mathbb{Z}^d$. s.t. $\mathcal{L}f_n = 0$. on B_n.

and $\sup_x |f_n(x)| \leq 1$. Set $g_n: \mathbb{R}^d \to \mathbb{R}^1$. by

$g_n(y) = f_n(ny)$ and extend by combi. Then:

$\exists (n_j)$. g harmonic on $\{|x| < 1\}$. s.t. $g_{n_j} \xrightarrow{a.c.c} g$.