Variations and Hölder

(1) **Def:** (E, d) metric space. For $x, y \in C([0,T], E)$.

i) $d_{\infty,[0,T]}(x, y) := \sup_{[0,T]} d(x_t, y_t)$.

ii) $|x|_{0,[0,T]} := \sup_{u,v} d(x_u, x_v)$

$|x|_{\infty,[0,T]} := d_{\infty,[0,T]}(x, 0)$. $0 \in E$ is const. (fix)

And denote $C_0([s,t], E) = \{ x \in C([s,t], E) \mid x(s) = 0 \}$.

iii) $C^{p\text{-var}}([0,T], E) = \{ x \in C \cap V^p \}$. where

V^p is set of p-variation functions.

equipped with $\| \cdot \|_{p\text{-var}, [s,t]} := \sup_{(t_i) \subset [s,t]} \left(\sum d(x_{t_i}, x_{t_{i+1}})^p \right)^{\frac{1}{p}}$

Def: i) Control $w : \Delta_{[0,T]} \to [0, \infty)$. is conti.

subadditive ($w(s,t) + w(t,u) \le w(s,u)$).

and zero on diagonal ($w(s,s) = 0 \ \forall s$)

ii) p-variation map x is controlled by a

control w if : $d(x_t, x_s)^p \le C\, w(s,t)$.

prop. $x \in C^{p\text{-var}}([0,T], E)$ controlled by control w.

$\Rightarrow \| x \|_{p\text{-var}, [s,t]} \le C\, w(s,t)^{\frac{1}{p}}$. $\forall s < t$.

prop. $x \in C^{p\text{-var}} \iff \overline{\lim_{\delta \to 0}} \sup_{\substack{(t_i) \subset [0,T] \\ \Delta t < \delta}} \sum d(x_{t_i}, x_{t_{i+1}})^p < \infty$.

prop. $(s,t) \longmapsto \|X\|^p_{p\text{-var},[s,t]}$ defines a control for $X \in C^{p\text{-var}}$.

Besides, it's additive. In particular, $t \longmapsto \|X\|_{p\text{-var},[0,t]}$ is conti. $\uparrow$.

Rmk: $p' \in [p,\infty) \longmapsto \|X\|_{p'\text{-var},[s,t]}$ is also conti. $\downarrow$. for $X \in C^{p\text{-var}}$.

Thm. (Time shift).

$X \in C([0,T],E)$. Then $X \in V^p \iff$ $\exists\, h \in C([0,T],[0,1])$. $\uparrow$ and $1/p$-Hölder path g. s.t. $X = g \circ h$.

Pf: Set $h(t) = \|X\|^p_{p\text{-var},[0,t]} / \|X\|^p_{p\text{-var},[0,T]}$.

Note $h(t_1) = h(t_2) \Rightarrow X_{t_1} = X_{t_2}$

So: $g \circ h = X$ exists.

easy to check $\|g\|_{1/p\text{-Höl},[0,1]} \le \|X\|_{p\text{-var},[0,T]}$.

Rmk: Note $\|g\|_{p\text{-var}} = \|X\|_{p\text{-var}}$. and

$\|g\|_{p\text{-var}} \le \|g\|_{1/p\text{-Höl}}$.

$\Rightarrow \|g\|_{p\text{-var}} = \|g\|_{1/p\text{-Höl}}$.

① Approxi. :

Def: $D = (t_i)$. partition of $[0,T]$. Set $X^D_t :=$

$$X_{t_i} + \frac{t - t_i}{t_{i+1} - t_i} X_{t_i, t_{i+1}} \quad \text{for } \forall\, t_i \le t \le t_{i+1}.$$

prop. $X \in C^{1\text{-var}} \Rightarrow \|X^D\|_{1\text{-var}} \le \|X\|_{1\text{-var}}$

Besides, $X^{D_n} \xrightarrow{n} X$. if $\Delta t^n \to 0$.

Lemma: $\overline{AC([0,T])}^{\|\cdot\|_{1\text{-var}}} = AC([0,T]) \subset V^1([0,T], E)$.

Pf: $\forall (X^n) \subset AC[0,T] \to X$. We prove

$X \in AC[0,T]$ as well.

Note $\sum |X_{t_i, s_i}| \le \sum |X^n_{t_i, s_i}| + \|X^n - X\|_{1\text{-var}}$

Cor. For $X \in C^{1\text{-var}}$. Then: $\|X^D - X\|_{1\text{-var}} \xrightarrow{|D| \to 0} 0$

$\Leftrightarrow X \in AC[0,T]$

For general p-var case. Next we need to introduce geodesic space:

Def: Metric space (E, d) is geodesic if $\forall a, b \in E$.

$\exists\, \gamma^{a,b} \in C([0,1], E)$. s.t. $\gamma^{a,b}(0) = a$. $\gamma^{a,b}(1) = b$.

and $d(\gamma^{a,b}_s, \gamma^{a,b}_t) = |t-s|\, d(a,b)$.

Rmk: i) $\gamma^{a,b}$ may not be unique.

ii) It means there're no shortcuts between a and b.

iii) $(S^1, \|\cdot\|_2) \subset \mathbb{R}^2$ isn't geodesic. But equip it with arclength $\Rightarrow$ it's!

Def: $D = (t_i) \subset [0,T]$. partition. $\tilde{X}^D_t := \gamma^{x_{t_i}, x_{t_{i+1}}}\left(\frac{t-t_i}{t_{i+1}-t_i}\right)$ for $t_i \le t \le t_{i+1}$ is geodesic approxi. for $X \in C$.

Lemma. If E is geodesic, $X \in C([0,T], E)$. Then $\tilde{X}^D \xrightarrow[|D|\to 0]{u} X$.

Pf: For $\forall t \in [t_i, t_{i+1}]$. $d(X_t, \tilde{X}^D_t) \le$

$d(X_t, X_{t_i}) + d(\tilde{X}^D_t, X_{t_i}) = d(X_t, X_{t_i}) + \left|\frac{t-t_i}{t_{i+1}-t_i}\right| d(X_{t_i}, X_{t_{i+1}})$.

By continuity. $d(X_t, \tilde{X}^D_t) \lesssim \varepsilon$.

Lemma. If E is geodesic. $X \in C^{p\text{-var}}([0,T], E)$. Then:

$\|\tilde{X}^D\|_{p\text{-var}} \le 3^{1-1/p} \|X\|_{p\text{-var}}$

Cor. By time-shift. it also holds for $C^{1/p\text{-Höl}}$.

Cor. Under conditions above. if $|D_n| \to 0$. Then:

$\tilde{X}^{D_n} \xrightarrow{u} X$ and $\sup_n \|\tilde{X}^{D_n}\|_{p\text{-var}} \le 3^{1-1/p} \|X\|_{p\text{-var}}$.

Rmk: By time-shift. it also for $C^{1/p\text{-Höl}}$.

Thm. (Compactness)

For $(X_n) \subset C([0,T], \mathbb{R}^d)$.

i) If $(X_n) \xrightarrow{u} X \in C([0,T])$. and $\sup_n \|X^n\|_{p\text{-var}}$ / $\sup_n \|X^n\|_{1/p\text{-Höl}} < \infty$. Then:

$X_n \to X$. in $\|\cdot\|_{p'\text{-var}}$ / $\|\cdot\|_{1/p'\text{-Höl}}$ $\forall p' > p$.

ii) If (X_n) equiconti. bdd. and $\sup_n \|X^n\|_{p\text{-var}}$ / $\sup_n \|X^n\|_{1/p\text{-Höl}} < \infty$. Then: $\exists (n_k) \subset \mathbb{Z}$. s.t.

$X^{n_k} \to X \in C^{p\text{-var}} / C^{1/p\text{-Höl}}$. in the way as i).

③ Representation:

Def: $C^{0,p\text{-var}} := \overline{C^\infty}^{\|\cdot\|_{p\text{-var}}}$.

Rmk: By closedness of $AC[0,T]$. we have:

$$C^{0,1\text{-var}} \subset AC \subsetneq C^{1\text{-var}}.$$

Lemma. $x \in AC[0,T] \Rightarrow \exists \dot{X} \in L^1$. s.t. $x_t = x_0 + \int_0^t \dot{x}_s\, ds$.

Pf: By Radon - Nikodym Thm.

prop. $L^1([0,T],\mathbb{R}^d) \xrightarrow{\varphi} C^{0,1\text{-var}}([0,T],\mathbb{R}^d)$ defines

$$y_t \longmapsto \int_0^t y_s \, ds + y_0 .$$

a isometric isomorphism.

Cor. $X \in C^{0,1\text{-var}} \iff \exists$ unique $\dot{X} \in L^1$. s.t.

$X = X_0 + \int_0^{\cdot} \dot{X}_t \, dt$. And $\|X\|_{1\text{-var}} = \|\dot{X}\|_{L^1}$

Pf: First consider in C^∞. check it satisfies conditions $\Rightarrow$ extend to L^1.

Rmk: Then we have: $C^{0,1\text{-var}} = AC[0,T]$

$\Rightarrow C^{0,1\text{-var}}$ is a polish space.

prop. $L^\infty([0,T],\mathbb{R}^d) \xrightarrow{\varphi} C^{1\text{-Höl}}([0,T],\mathbb{R}^d)$ defines

$$y \longmapsto y_0 + \int_0^{\cdot} y_t \, dt$$

a isomorphic isomorphism.

So. $X \in C^{1\text{-Höl}} \iff \exists \dot{X} \in L^\infty$ unique. st.

$X_t = X_0 + \int_0^t \dot{X}_s \, ds$. Besides. $\|X\|_{1\text{-Höl}} = \|\dot{X}\|_\infty$.

Pf: Similarly. by extension.

Rmk: i) Time-change function η of $X \in C^{1\text{-var}}$

st. $\eta \circ \phi = X$. as Thm in (1) satisfies:

$|\dot{\eta}(t)| \equiv \|X\|_{1\text{-var}} = \|\eta\|_{1\text{-Hölder}}$.

Since $\|\eta\|_{1\text{-var},[0,\phi(t)]} = \int_0^{\phi(t)} |\dot{\eta}(s)|\,ds$

$= \|X\|_{1\text{-var},[0,T]} \cdot \phi(t) = \|X\|_{1\text{-var},[0,t]}$

$\Rightarrow \eta$ has const. speed.

ii) $\overline{C^\infty}^{\|\cdot\|_{1\text{-Höl}}} = C^1([0,T],\mathbb{R}^d)$.

Next. we consider more general Sobolev space:

Lemma. i) $1 \le p \le \infty$. $W^{1,p}([0,T],\mathbb{R}^d) = \{ X = X_0 + \int_0^\cdot \eta_t\,dt :$ $\eta \in L^p \}$.

ii) $p \in (1,\infty)$. $X \in W^{1,p}([0,T],\mathbb{R}^d) \Leftrightarrow \sup_{(t_i)} \sum \frac{|X_{t_i,t_{i+1}}|^p}{|t_{i+1}-t_i|^{p-1}} \sim \lim_{\delta \to 0} \sup_{\substack{(t_i)\\ |\Delta t| < \delta}} \sum \frac{|X_{t_i,t_{i+1}}|^p}{|t_{i+1}-t_i|^{p-1}} < \infty$.

Rmk: $W^{1,1} = AC[0,T]$. $W^{1,\infty} = C^{1\text{-Höl}}$

Pf: ii) By Hölder ($\Rightarrow$) is trivial.

Conversely. By condition $\Rightarrow X \in AC[0,T]$.

So. $\exists X^{D_n} \to X$ in $\|\cdot\|_{1\text{-var}}$.

Besides $\dot{X}^{D_n} = \sum X_{t_i,t_{i+1}} I_{(t_i^n, t_{i+1}^n]}$. Use Fatou's.

Lemma For $p \in (1,\infty)$. $X \in W^{1,p}([0,T],\mathbb{R}^d)$. $\|X\|_{W^{1,p},[s,t]} (t-s)^{1-1/p}$

is control function for X.