ODEs.

For $V = (V_1 \cdots V_d) \in \mathbb{R}^{e \times d}$. $x \in C^{1\text{-var}}([0,T], \mathbb{R}^d)$, and $y_0 \in \mathbb{R}^e$. We denote $\pi_V(0, y_0; x)$ is solution of $dy_t = V(y_t)\, dx_t$. $y(0) = y_0$. $(*)$

(1) <u>Lemma</u> (Existence)

If V is bdd. conti. Then there exists solutions of $(*)$. s.t.

$\| \pi_V(0, y_0; x) \|_{1\text{-var}, [s,t]} \leq \|V\|_\infty \|x\|_{1\text{-var}, [s,t]}$.

<u>Rmk</u>: i) Solution of $(*)$ may not be unique.

ii) Remove "bdd". There exists a explosion time τ for $\pi_V(0, y_0; x)_t$.

<u>Pf</u>: As common, by Euler approxi. y^{D_n}.

Then check Ascoli Thm holds to have y.

<u>Lemma</u> If V is linear growth. i.e. $\exists A > 0$. s.t. $|V_i(x)| \leq A(1 + |x|)$. Then explosion time τ doesn't exist. Besides:

$\|y\|_{\infty, [0,T]} \leq (|y_0| + A \|x\|_{1\text{-var}, [0,T]})\, e^{A \|x\|_{1\text{-var}, [0,T]}}$.

Thm (Uniqueness)

If V is Lipschitz conti. $\tilde{V} := \sup \frac{|V(z) - V(y)|}{|z-y|}$

$\ell := \tilde{V} \|X\|_{1\text{-var},[0,T]}$. Then the solution of (*)

is unique and:

$$\|Z_V(0,y_1;X) - Z_V(0,y_2;X)\|_{\infty,[0,T]} \leq |y_1 - y_2| e^{\ell}.$$

$$\| \sim \|_{1\text{-var},[0,T]} \leq |y_1 - y_2| e^{2\ell} \ell.$$

Pf: By Gronwall's inequi.

(2) Time-change:

Next, we assume existence and uniqueness hold.

prop. If $\phi \in C([0,T],[0,\tilde{T}])$, $\uparrow$. Then:

$$Z_V(0,y_0;x\circ\phi)_t = Z_V(0,y_0;X)_{\phi(t)} \text{ on } [0,\tilde{T}].$$

Pf: Change of variable in $\int$. By unique.

Def: i) $X \in C([0,T])$, $\tilde{X} \in C([T,u])$. Concatenation

is defined by $X \sqcup \tilde{X}(t) = \begin{cases} X_t, & t \leq T \\ X_T + \tilde{X}_t - \tilde{X}_T, & t > T. \end{cases}$

ii) Time-reverse $\overleftarrow{X}^T$ is defined by:

$\overleftarrow{X}^T : t \mapsto X_{T-t}$ for $X \in C[0,T]$.

prop. i) $Z_V(0,\eta_0; X \sqcup \tilde{X}) = \begin{cases} Z_V(0,\eta_0;X) & [0,T] \\ Z_V(0, Z_V(0,\eta_0;X)_T ; \tilde{X}) & [T,u]. \end{cases}$

ii) $Z_V(0,\eta_T; \overleftarrow{X}^T)_{T-t} = \eta_t$.

Pf: ii) Check η_{T-t} is solution of:

$$\tilde{\eta}_t = \eta_T + \int_0^t V(\tilde{\eta}_s)\, dx_{T-s}$$

$$(\Leftrightarrow \int_{T-t}^T V(\eta_s)\, dx_s = \eta_T - \eta_{T-t}).$$

Cor. Reparametrize $\overleftarrow{X}^T$ on $[T, 2T]$. i.e. $\overleftarrow{X}_t = X_{2T-t}$.

Then: $Z_V(0,\eta_0; X \sqcup \overleftarrow{X})_{2T} = \eta_0$.

Pf: Combine i) and ii) above.

(3) Regularity of $Z_V(0,\cdot;\cdot)$:

Thm. (Continuity)

For $V^1, V^2 \in C^{Lip}(\mathbb{R}^{e\times d})$. Set $\tilde{V} \triangleq \|V^1\|_{Lip} \vee \|V^2\|_{Lip}$.

$X^1, X^2 \in C^{1\text{-var}}$. $l \triangleq \|X^1\|_{1\text{-var},[0,T]} \vee \|X^2\|_{1\text{-var},[0,T]}$.

$\Rightarrow \|Z_{V^1}(0,\eta_0^1;X^1) - Z_{V^2}(0,\eta_0^2;X^2)\|_{\infty,[0,T]} \le (|\eta_0^1 - \eta_0^2| + \tilde{V}\|X^1 - X^2\|_{1,[0,T]} + \|V^1 - V^2\|_\infty l)\, e^{2\tilde{V}l}$.

Pf: Estimate $|\eta_t^1 - \eta_t^2|$. by Gronwall's.

__Cor.__ Under conditions above. We have:

$$\| Z_{V^1}(\cdot\cdot) - Z_{V^2}(\cdot\cdot) \|_{1\text{-var},[0,T]} \le 2(|y_0^1 - y_0^2| V\ell + V\|X^1 - X^2\|_{1\text{-var}} + \|V^1 - V^2\|_\infty \ell)\, e^{3V\ell}.$$

__Pf:__ Estimate $|y_{st}^1 - y_{st}^2|$.

To consider the smoothness of map $Z_V(\cdot\cdot)$.

We assume V satisfies non-explosive condition:

if $\forall R > 0.\ \exists M > 0.$ s.t. $\|X\|_{1\text{-var}} + |y_0| \le R \Rightarrow \|Z_V(\cdot\cdot)\|_\infty \le M$.

__Thm.__ (Smoothness)

For $X \in C^{1\text{-var}}$. $V \in C^k_{loc}$. non explosive. Then:

$(y_0, x) \in \mathbb{R}^e \times C^{1\text{-var}} \mapsto Z(0, y_0; x) \in C^{1\text{-var}}$ is

C^k-diff. in sense of Fréchet.

__Rmk:__ By Duhamel's principle. We have

direct representation of derivative