Algebra and Signatures

Next, we fix $E = \mathbb{R}^d$, on $[0,T]$. We reuse the notations in "low regularity" section before.

(1) **Properties**:

Def: i) N^{th}-signature of $Y \in C^{1\text{-var}}$ is given by:

$$S_N(Y)_{s,t} \triangleq \left(1, \int_s^t dY_u, \int_{\Delta^2_{[s,t]}} dY_u \otimes dY_v, \cdots, \underbrace{\square}_{N^{th}}\right)$$

$t \mapsto S_N(Y)_{s,t}$ is called N^{th}-lift of Y.

ii) $\|g\|_{T^N} := \max_{0 \le k \le N} \|\pi_k(g)\|_{L^2}$.

prop. $X \in C^{BV}$. Then: $dS_N(X)_{s,t} = S_N(X)_{s,t} \otimes dX_t$, $S_N(X)_{s,s} = 1$.

Pf: $\pi_k(S_N(X)_{s,t}) = \int_s^t \left(\int_{\Delta^k_{[s,r]}} dX_{r_1} \otimes \cdots dX_{r_{k-1}}\right) dX_r$

$$= \int_s^t \pi_{k-1}(S_N(X)_{s,r}) \otimes dX_r$$

Set the k^{th}-tensor product $= 0$ $\forall k > N$. We have

$$S_N(X)_{s,t} = 1 + \int_s^t S_N(X)_{s,r}\, dX_r.$$

Lemma. $S_N(X)_{\varphi(s),\varphi(t)} = S_N(X^\varphi)_{s,t}$, for φ conti. $\uparrow$.

Pf: Note $S_N(X)_{s,t} = \pi_{S_N(X)_{s,\cdot}}(s, 1; X_r)_t$

Thm. $\gamma \in C^{1\text{-var}}[0,T]$. $\eta \in C^{1\text{-var}}[T,2T]$. Then:

$$S_N(\gamma \cup \eta)_{0,2T} = S_N(\gamma)_{0,T} \otimes S_N(\eta)_{T,2T}.$$

In particular. $S_N(x)_{s,u} = S_N(x)_{s,t} \otimes S_N(x)_{t,u}$.

Pf: By induction on N. $N=0$ is trivial.

$$S_{N+1}(x)_{s,u} = 1 + \int_s^u S_N(x)_{s,r}\, dx_r \quad (\text{truncated})$$

$$\overset{\text{hypo.}}{=} 1 + \int_s^t S_N(x)_{s,r}\, dx_r + S_{N+1}(x)_{s,t} \otimes \left(\int_t^u S_N(x)_{t,r}\, dx_r\right).$$

$$= S_{N+1}(x)_{s,t} \otimes \left(1 + (S_{N+1}(x)_{t,u} - 1)\right).$$

prop. $X \in C^{1\text{-var}}[0,T]$. Then: $S_N(X)_{0,T} \otimes S_N(\overleftarrow{X})_{0,T} = 1$.

Pf: By Thm above and $t \mapsto S_N(x)_{0,t}$ is the Solution of ODE with $V_\cdot = S_N(x)_{0,\cdot}$

Apply Time-change results of solution

prop. (X_n) bdd in $C^{1\text{-var}}[0,T]$. and $X_n \xrightarrow{u} X \in C^{1\text{-var}}$.

Then: $S_N(X_n)_t \xrightarrow{u} S_N(X)$. $\forall t$.

Pf: By continuity of $\pi_V(0, y_0; x)$.

(2) Lie group / algebra:

Def: Lie group is a group and also a smooth manifold. The group operations are smooth maps.

Rmk: $(T_1^N(\mathbb{R}^d), \otimes)$ is a Lie group.

Next, we equip it with metric $\rho(\cdot,\cdot)$ define by $\rho(g,h) = \max_{1\le i\le N} |\pi_i(g-h)|$.

Lemma. $\{g_n\}, g \in T_1^N(\mathbb{R}^d)$. Then $\rho(g_n, g) \to 0 \iff \rho(g_n^{-1}\otimes g, 1) \to 0$.

Pf: Note the group operation $(\cdot)^{-1}, \otimes$ are conti.

Def: $(V, +, \cdot)$ LS equip with $\begin{array}{ccc} V\times V & \longrightarrow & V \\ (g,h) & \longmapsto & [g,h] \end{array}$ is Lie algebra if $[\cdot,\cdot]$ satisfies Jacobi id.

Rmk: $(T_0^N(\mathbb{R}^d), \otimes)$ is Lie algebra with $[g,h] := g\otimes h - h\otimes g$.

Note that $e^a \otimes e^b \ne e^{a\otimes b}$. e.g. for $N=2$.

$$LHS = \left(1+a+\frac{a^{\otimes 2}}{2}\right)\left(1+b+\frac{b^{\otimes 2}}{2}\right) = e^{a+b+\frac{1}{2}[a,b]}.$$

generally. $e^a \otimes e^b = e^{a+b+\frac{1}{2}[a,b]+\frac{1}{12}[a,[a,b]]+\cdots}$.

Thm. (Campbell - Baker - Hausdorff)

Define $\mathrm{ad}_a(b) := [a,b]$. For $a, b \in T_0^N(\mathbb{R}^d)$,

$$\log(e^a \otimes e^b) = b + \int_0^1 H(e^{t\,\mathrm{ad}_a} \circ e^{\mathrm{ad}_b})\, a \, dt.$$

Where $H(z) \overset{\Delta}{=} \frac{\ln z}{z-1} = \sum_{n \geq 0} (-1)^n (z-1)^n / n+1$

Rmk: We have $\log(e^a \otimes e^b) \in \mathcal{L}^N(\mathbb{R}^d)$. (Recall

$\mathcal{L}^N \overset{\Delta}{=} [\mathcal{L}^{N-1}, \mathbb{R}^d]$. $\mathcal{L}^1 = \mathbb{R}^d$.) if $a, b \in \mathcal{L}^N$.

Thm. (Chow's)

For $g \in e^{\mathcal{L}^N(\mathbb{R}^d)}$. Then $\exists (v_k)_1^m \in \mathbb{R}^d$. s.t.

$$g = e^{v_1} \otimes e^{v_2} \cdots \otimes e^{v_m}.$$

Cor. $\exists X : [0,1] \to \mathbb{R}^d$. piecewise lin. s.t. g

$= S_N(X)_{0,1}$. $g \in e^{\mathcal{L}^N(\mathbb{R}^d)}$.

Pf: Note for $t \in [0,1] \mapsto ta$. $a \in \mathbb{R}^d$.

$$S_N(t \mapsto \cdots)_{0,1} = 1 + \sum_1^N a^{\otimes k} \int_{0<t_1 \cdots < t_k < 1} dt_1 \cdots dt_k.$$

$$= e^a \text{ in } T^N(\mathbb{R}^d)$$

So set $X = X_1 \sqcup X_2 \cdots \sqcup X_m$. and dilate it.

$$\Rightarrow S_N(X)_{0,1} = e^{v_1} \otimes \cdots \otimes e^{v_m}.$$

Cor. $(g_k) \subset e^{\mathcal{L}^n(\mathbb{R}^d)} \to 1$. Set x_k is the piecewise linear path of g_k. Then: the length of $x_k \to 0$. ($\int_0^1 |dx_k| \to 0$)

(3) **Free Nilpotent group:**

prop. $G_N(\mathbb{R}^d) := \{ S_N(x)_{0,1} \mid x \in C^{1\text{-var}}[0,1] \}$ (FNG)

$= e^{\mathcal{L}^n(\mathbb{R}^d)} = \langle e^{\mathbb{R}^d} \rangle \overset{\Delta}{=} \{ \bigotimes_1^m e^{v_i} \mid m \geq 1, v_i \in \mathbb{R}^d \}$.

Pf: By Chow's Thm.

Thm. (Geodesic existence)

For $g \in G_N(\mathbb{R}^d)$. Carnot - Carathéodory norm is $\|g\| := \inf \{ \int_0^1 |dy| : y \in C^{1\text{-var}}[0,1], S_N(y)_{0,1} = g \}$.

$\|g\|$ is finite and can be achieved at some $y^* \in C^{1\text{-var}}[0,1]$. s.t. $S_N(y^*)_{0,1} = g$.

Rmk: We find the shortest path to have the correct signature.

Pf: $\exists\, y^n$ s.t. $\int_0^1 |y^n| \to \|g\|$. $\Rightarrow \|y^n\|_{1\text{-var}} \leq C$ $\forall n$.

reparametrize y^n. s.t. $\|y^n\|_{1\text{-Höl}} = \|y^n\|_{1\text{-var}}$

So: By Ascoli's Lemma. $\exists\, y^{n_k} \xrightarrow{u} y^*$.

By continuity of $S_N(\cdot)$ and Fatou's. ✓.

Cor. i) $\|g\|=0 \iff g=1$ ii) $\|g\| = \|g^{-1}\|$.

iii) $\|g \otimes h\| \le \|g\| + \|h\|$.

iv) $g \mapsto \|g\|$ is conti.

Pf: ii) Note $g^{-1} = S_N(\overleftarrow{\gamma}_g^*)_{0,1}$

iii) $g \otimes h = S_N(X_g \sqcup X_h)_{0,1}$

iv) $g_n \xrightarrow{\|\cdot\|_{T^N}} g \iff g_n^{-1} \otimes g \to 1$

So: $\|g_n^{-1} \otimes g\| \to 0$. By i).

with last cor. in (2).

Cor. $d(g,h) := \|g^{-1} \otimes h\|$ induce a metric called C-C metric.

satisfies: i) conti.

ii) $\|g^{-1} \otimes h\| = \|g \otimes h^{-1}\|$.

Rmk: All homogeneous norms on G^N are equi. (e.g. $\sup_k |\pi_k(\cdot)|^{1/k} \sim \|\cdot\|$)

Cor. i) $(G^N(\mathbb{R}^d), d)$ is polish space st. $\forall$ closed bdd sets are cpt.

ii) $(G^N(\mathbb{R}^d), d)$ is geodesic space.

And $\gamma_t^{g,h} = g \otimes S_N(\gamma^*_{g^{-1}\otimes h})_{0,t}$ is the connecting geodesics.

Pf: ii) $d(\gamma_s, \gamma_t) = \| \gamma_s^{-1} \otimes \gamma_t \|$

$$= \| S_N(\gamma^*_{g,h})_{s,t} \|$$

$$\leq \int_0^1 |d\gamma^*_{\square}| \quad (t-s)$$

$$= (t-s)\, d(g,h)$$

If $d(\gamma_s, \gamma_t) < (t-s)\, d(g,h)$

$$\underset{\text{contrad.}}{\Rightarrow} d(g,h) \leq d(\gamma_0, \gamma_s) + d(\gamma_s, \gamma_t) + d(\gamma_t, \gamma_1)$$

$$< (s + (t-s) + (1-t))\, d(g,h)$$

prop. $x \in C^{1\text{-var}}[0,T] \Rightarrow \| S_N(x) \|_{1\text{-var}, [0,T]} = \| x \|_{1\text{-var}, [0,T]}$.

Pf: i) $\| S_N(x)_{s,t} \| = | \int_s^t |d\gamma^*| | \geq | \int_s^t d\gamma^* |$

$$\overset{\text{Def.}}{=} | x_{s,t} | \ (= | \gamma^*_{s,t} |).$$

So: $d(S_N(x)_s, S_N(x)_t) \leq d(x_s, x_t)$.

ii) $\| S_N(x)_{s,t} \| \leq \int_s^t |dx| = \| x \|_{1\text{-var}, [s,t]}$

(4) Hölder and Variation:

Next we use $\| \cdot \| = \max_k |z^k(\cdot)|^{\frac{1}{k}}$ to define α-Hölder and p-Varn.

Thm. For $p \geq 1$. $C^{1/p\text{-Höl}}([0,T], G^N(\mathbb{R}^d))$ and

$C^{p\text{-var}}([0,T], G^N(\mathbb{R}^d))$ are complete, nonseparable.

prop. $\forall X \in C^{p\text{-var}}([0,T], G_N(\mathbb{R}^d))$. $p \geq 1$. $\exists (X^n) \subset C^{1\text{-Hölder}}([0,T], \mathbb{R}^d)$,

s.t. $d_{\infty,[0,T]}(X, S_N(X^n)) \to 0$. and $\sup \|S_N(X^n)\|_{p\text{-}\omega} < \infty$

Pf: By property of geodesic space.

Thm. (Ascoli)

$(X^n) \subseteq C([0,T], G_N(\mathbb{R}^d))$. bdd. equiconti.

and $\sup_n \|X^n\|_{p\text{-var}}$ / $\sup \|X^n\|_{1/p\text{-Hölder}}$. $\Rightarrow$

$\exists$ subseq s.t. $X^{n_k} \to X \in C^{p\text{-var}} / C^{1/p\text{-Höl}}$.

in $\|\cdot\|_{p_1\text{-var}}$ / $\|\cdot\|_{1/p_1\text{-Höl}}$. $\forall p_1 > p$.

Pf. property of Hölder / Variation space.