Enhanced Brownian Motions

(1) Lévy area:

Def: i) Given two indept BMs $\beta_t, \tilde{\beta}_t$. Lévy area of them is $t \in [0,\infty) \mapsto \frac{1}{2}\int_0^t \beta_s d\tilde{\beta}_s - \tilde{\beta}_s d\beta_s$ in sense of Itô integral.

Rmk: i) BM has finite 2V but has infinite p-variation for $p \le 2$.

ii) It makes no difference to use Stratonovich integral.

ii) For d-dim BM $B_t = (B^1 \cdots B^d)$. Set $A = (A_{ij})_{d\times d}$, defined by $A^t_{ij} \stackrel{\Delta}{=} \frac{1}{2}(\int_0^t B^i_t dB^j_t - \int_0^t B^j_t dB^i_t)$.

increment $A^{s,t}_{i,j}$ is:

$A^t_{ij} - A^s_{ij} - \frac{1}{2}(B^i_s B^j_{s,t} - B^j_s B^i_{s,t}) = \frac{1}{2}(\int_s^t B^i_{s,r} dB^j_r - \int_s^t B^j_{s,r} dB^i_r)$

Rmk: i) $A^{s,t}_{ij} \neq A^t_{ij} - A^s_{ij}$ is for retaining the path in $e^{\mathbb{R}^d}$.

But $A_t = A_{0,t}$

ii) $A_{s,t} \in so(d) \cong [\mathbb{R}^d, \mathbb{R}^d]$. in fact.

Lemma. $\forall \lambda > 0.$ $A_{\lambda t} \overset{d}{\sim} \lambda A_t.$ $A_{s,t} \overset{d}{\sim} A_{0,t-s}.$

prop. (exponential integrability).

$\exists \eta > 0.$ s.t. $\mathbb{E}(e^{\eta |A^{ij}_{s,t}| / |t-s|}) < \infty.$ $\forall i,j.$

Pf: **Lemma.** $\forall \eta < \frac{1}{2}.$ $\mathbb{E}(e^{\eta |B|^2_{\infty,[0,T]} / T}) < \infty$

Pf: $LHS = \int_0^\infty \frac{2x\eta}{T} e^{\frac{\eta x^2}{T}} \mathbb{P}(|B|_\infty \geq x) dx$

By reflection principle :

$\mathbb{P}(|B|_\infty \geq x) = 4 \mathbb{P}(B_T \geq x)$

Note $|A^{ij}_{s,t}| \sim |t-s| \cdot A^{ij}_1$ and

$\int_0^1 \beta_s d\tilde{\beta}_s \mid \mathcal{F}^\beta \sim N(0, \int_0^1 \beta_s^2 ds) \overset{\Delta}{=} Z$

$LHS = \mathbb{E}(\mathbb{E}(e^{\eta |\int_0^1 \beta_s d\tilde{\beta}_s|} \mid \mathcal{F}^\beta))$

$= \mathbb{E}(\mathbb{E}(e^{\eta |Z|} \mid \mathcal{F}^\beta)) \leq 2 \mathbb{E}(\mathbb{E}(e^{\eta Z} \mid \mathcal{F}^\beta))$

$= 2 \mathbb{E}(e^{\eta^2 \int_0^1 \beta_s^2 ds / 2})$

$\leq 2 \mathbb{E}(e^{\eta^2 |\beta|_\infty / 2}) \overset{Lem}{<} \infty$

Thm (Time-change expression)

B_t is d-dim BM. Let $\beta = B_i.$ $\tilde{\beta} = B_j.$ $i \neq j.$

$A_t = \int_0^t \beta_s d\tilde{\beta}_s - \cdots.$ $a(t) := \frac{1}{4} \int_0^t (\beta_s^2 + \tilde{\beta}_s^2) ds$

$\Rightarrow A(a^{-1}(t)) \sim$ 1-dim BM. indept of $\beta_s^2 + \tilde{\beta}_s^2.$

Pf: Denote $r_t = (\beta_t^2 + \tilde{\beta}_t^2)^{\frac{1}{2}}$. $Y_t = \int_0^t \frac{\beta_s}{r_s} d\beta_s + \int_0^t \frac{\tilde{\beta}_s}{r_s} d\tilde{\beta}_s$

Apply Itô's on $\beta_t^2 + \tilde{\beta}_t^2 / 2$. we have:

$$\frac{r_t^2}{2} = \int_0^t r_s dY_s + t \quad (*)$$

Note $[A]_t = \frac{1}{4}\int_0^t r_s^2 ds$. $[Y]_t = t$. $\langle Y, A\rangle_t = 0$.

Besides. r_t is unique solution of SDE $(*)$.

$\Rightarrow \sigma(r_s, s \le t) \subset \sigma(Y_s, s \le t) \Rightarrow A_t \perp r_t$.

(2) Enhanced BMs:

① Def: B is d-dim BM and A is its Lévy area.

$\mathbb{B}_t := e^{B_t + A_t}$. conti. $G^2(\mathbb{R}^d)$-valued process

is called enhanced BM.

Rmk: Set $\mathbb{B}_{s,t} = \mathbb{B}_s^{-1} \otimes \mathbb{B}_t$. it's consistent

with $\mathbb{B}_{s,t} = e^{B_{s,t} + A_{s,t}}$.

prop. i) $\mathbb{B}_0 = 1$. $\forall w \in \Omega$. ii) $t \mapsto \mathbb{B}_t(w)$ is conti. $\forall w$.

iii) $\mathbb{B}_{t,t+h} = \mathbb{B}_t^{-1} \otimes \mathbb{B}_{t+h}$ indept of $\sigma(\mathbb{B}_s, s \le t)$.

iv) $(\mathbb{B}_{s,s+t})_t \overset{d}{\sim} (\mathbb{B}_t)_t$.

Pf: i). ii) are trivial. iii) $\sigma(\mathbb{B}_s, s \le r) = \sigma(B_s, s \le r)$

iv) $(A^{ij}_{s,s+t}) = \frac{1}{2}\left(\int_s^{s+t} B^i_{s,r} dB^j_r - \cdots\right)$

$= \frac{1}{2}\left(\int_s^{s+t} B^i_{s,r} dB^j_{s,r} - \cdots\right)$

$$\sim \frac{1}{2}\left(\int_0^t B_r^i \wedge B_r^j - B_r^j \wedge B_r^i\right) = (A_t^{i,j})$$

Lemma. $\delta_\lambda : G_N(\mathbb{R}^d) \to G_N(\mathbb{R}^d)$. We have:

$$(x_1, x_2 \cdots x_N) \mapsto (\lambda x_1, \lambda^2 x_2 \cdots \lambda^N x_N)$$

$$(\mathbb{B}_{\lambda^2 t})_t \overset{d}{\sim} (\delta_\lambda \mathbb{B}_t)_t . \quad \delta_\lambda \text{ def on } G^2$$

Pf: $(B_{\lambda t}, A_{\lambda t}) \overset{d}{\sim} (\lambda^{\frac{1}{2}} B_t, \lambda A_t)$.

② Regularity:

Recall BM is p-Hölder and has finite 1/p-var only when $p \in (0, \frac{1}{2})$. Next, we want to show $\mathbb{B}_t$ is α-Höl geometric rough path. $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

Lemma. $d(\mathbb{B}_s, \mathbb{B}_t) = \| e^{B_{s,t} + A_{s,t}} \| \sim |B_{s,t}| \vee |A_{s,t}|^{\frac{1}{2}}$

Pf: recall by equi. of homo. norm:

$$\|g\|_{c.c} \sim \max_{1 \le i \le N} |Z_i(\log g)|^{\frac{1}{i}}.$$

Thm. $\exists \eta > 0$, s.t. $\sup_{s \in [0,T]} \mathbb{E}\left(e^{\eta \|\mathbb{B}_{s,t}\|^2 / |t-s|}\right) < \infty$.

Pf: $\|\mathbb{B}_{s,t}\|^2 \sim |t-s| \, \|\mathbb{B}_1\|^2$.

$\|\mathbb{B}_1\|^2 \sim |B_1| + |A_1|^2$. by integrability of gaussian.

Cor. i) $\forall \alpha \in [0, \frac{1}{2})$, $\exists \eta > 0$. s.t. not depend T.

$$\mathbb{E}(e^{\eta \|B\|^2_{\alpha\text{-Höl}, [0,T]} / T^{1-2\alpha}}) < \infty .$$

ii) $\|x\|_{\varphi\text{-Höl}, [0,T]} := \sup_{s,t} d(x_s, x_t) / \varphi(t-s)$.

For $\varphi(h) = \sqrt{h \log 1/h}$. We have:

$\exists \eta = \eta(h)$. s.t. $\mathbb{E}(e^{\eta \|B\|^2_{\varphi\text{-Höl}, [0,T]}}) < \infty$.

Pf: i) $\|B\|^2_{\alpha\text{-Höl}, [0,T]} / T^{1-2\alpha} \sim \|B\|^2_{\alpha\text{-Höl}, [0,1]}$

Cor. $\mathbb{E}(\|B\|^q_{\alpha\text{-Höl}})^{\frac{1}{q}} \sim q^{\frac{1}{2}}$. $\alpha \in (0, \frac{1}{2})$

Pf: expand $\exp(x)$ in i).

Rmk: It also holds in norm $\|x\|_{\varphi\text{-var}}$

$:= \inf \{ M > 0 \mid \sup_{D} \sum_{[t_i, t_{i+1}] \subset D} \varphi(d(x_s, x_t) / M) \leq 1 \}$

$\varphi(h) \triangleq h^2 / \ln^* \ln^* 1/h$. $\ln^* \triangleq 1 \vee \ln$

Prop. (law of iterated logarithm)

For $\varphi(h) = (h \ln^* \ln^* 1/h)^{\frac{1}{2}}$. $\exists$ const. $c > 0$. s.t.

$$\mathbb{P}(\overline{\lim_{h \to 0}} \|B\|_{\infty, [0,h]} / \varphi(h) = c) = 1 .$$

Pf: By Rmk above. $L \triangleq \overline{\lim_{h \to 0}} \|B\|_{\infty, [0,h]} / \varphi(h) < \infty$.

Besides. $L \geq \overline{\lim}_{n\to\infty} |B_n| / \varphi(n) = \sqrt{2} > 0$.

Note $\sigma(\mathbb{B}_t) = \sigma(B_t)$. by 0-1 law.

and $t B_{\frac{1}{t}}$ is BM. again. $\Rightarrow L \equiv$ const.

(3) Approxi. :

① Geodesics approxi.

Recall $\mathbb{B}_t \overset{a.s.}{\in} C_0^{1,\alpha\text{-Höl}}([0,T], G_2(\mathbb{R}^d))$. I.e. $\mathbb{B}_t$ is

$d_{\alpha\text{-Höl}}$ - limit of lift of smooth path. $\alpha < \frac{1}{2}$.

Rmk: i) The approxi. depends on deterministic facts

then applys on every $w \in \Omega$. almost.

ii) It also relys on the informations of

$A(w)$ and $\beta(w)$.

② Piecewise linear approxi.

Set (D_n) is nested partition. i.e. $D_n \subset D_{n+1}$.

and $|D_n| \to 0$ $(n \to \infty)$. $\mathcal{F}_n := \sigma(B_t \mid t \in D_n)$.

Define : $B^n \overset{\Delta}{=} B^{D_n}$. $\mathbb{B}^n := S_2(B^n)$.

Prop. $\forall$ fix $t \in [0,T]$. $\mathbb{B}_t^n \to \mathbb{B}_t$. in L^2 / a.s.

Pf: i) Recall $\mathbb{E}(B_t \mid B_T) = \frac{t}{T} B_T$. $\forall t \le T$.

$\Rightarrow \mathbb{E}(B_t \mid \mathcal{G}_n) = B_t^n \to B_t$

follows from Lévy Thm. and converge property of gaussian.

ii) Denote $\beta = B_i$. $\tilde{\beta} = B_j$. $i \ne j$.

$\mathbb{E}(\int_0^t \beta \, d\tilde{\beta} \mid \mathcal{G}_n) = \lim_n \mathbb{E}(\sum_{D_n} \beta_{t_i} \tilde{\beta}_{t_i, t_{i+1}} \mid \mathcal{G}_n)$

$\overset{\beta \perp \tilde{\beta}}{=} \lim_n \sum_{D_n} \beta_{t_i}^n \tilde{\beta}^n_{t_i, t_{i+1}} = \int_0^t \beta^n \, d\tilde{\beta}^n$.

$\Rightarrow \mathbb{E}(A_t \mid \mathcal{G}_n) = A_t^n \to A_t$.

Thm (Uniform bdd)

$\forall \alpha \in [0, \frac{1}{2})$. $\exists M$ r.v. ≥ 0. with Gaussian tail.

s.t. $\sup_{1 \le k \le \infty} \| \mathbb{B}^n \|_{\alpha\text{-Höl}, [0,T]} \le M$. where $\mathbb{B}_\infty = \mathbb{B}$.

Pf: Note we have: $\mathbb{E}(B_{s,t} \mid \mathcal{G}_n) = B_{s,t}^n$. and

$\mathbb{E}(A_{s,t} \mid \mathcal{G}_n) = A_{s,t}^n$.

$|A^{i,j}_{s,t}| \le |A_{s,t}| \le \|\mathbb{B}_{s,t}\|^2 \le M_1 (t-s)^{2\alpha}$.

where $M_1 = \| \mathbb{B} \|_{\alpha\text{-Höl}, [0,1]}$. has gaussian tail. And $\| M_1 \|_{L^2} = C^{\frac{1}{2}}$.

$\Rightarrow \mathbb{E}(|A^{i,j}_{s,t}| \mid \mathcal{G}_n) \le \sup_n \mathbb{E}(M_1 \mid \mathcal{G}_n) (t-s)^{2\alpha}$.

By Doob's inequi. we also have:

$\|M_2\|_{L^2} \sim q^{\frac{1}{2}}$. $M_2 := \sup_n \mathbb{E}(m_\cdot | \mathcal{F}_n)$.

So: $\|\tilde{B}^n_{s,t}\| \le (t-s)^\tau M_2$.

Set $M = M_1 + M_2$. We have conclusion

<u>Cor.</u> $\rho_{\tau\text{-Höl},[0,T]}(S_2(B^{D_n}), \mathbb{B}) \to 0$.

a.s. / in L^p. $\forall p \ge 1$.

<u>Pf:</u> By interpolation for L^p case.

<u>Rmk:</u> It also holds for general dissection

D. st. $|D| \to 0$. i.e. $\forall \eta \in (0, \frac{1}{2} - \tau)$

$|\rho_{\tau\text{-Höl}}(S^2(B^n), \mathbb{B})|_{L^2} \le C q^{\frac{1}{2}} |D|^{\frac{\eta}{2}}$

<u>Pf:</u> First estimate $Z_k(\cdots)$.

③ <u>Weak convergence.</u>

Recall for (ξ_i) indept RW in $\mathbb{R}^d$. st. $\xi_i \sim \xi$.

with $\mathbb{E}(\xi \otimes \xi) = I$. We have:

<u>Thm.</u> (Donsker)

$$W_t^{(n)} = (\xi_1 + \cdots \xi_{[nt]} + (nt - [nt])\xi_{[nt]+1}) / n^{\frac{1}{2}}$$

$$\xrightarrow{w} \text{SBM. on } (C^0([0,1], \mathbb{R}^d), \|\cdot\|_\infty).$$

Cor. (Lamperti)

If additionally $\mathbb{E}(|\xi|^{2p}) < \infty$, the convergence can hold in α-Hölder topology s.t. $\alpha < (p-1)/2p$.

We have similar Thm for EBMs:

Thm. $(\xi_i) \overset{i.i.d}{\sim} \xi$. RW in $\mathbb{R}^d$. s.t. $\mathbb{E}(\xi) = 0$. $\mathbb{E}(|\xi|^p) < \infty$. for $\forall p \geq 1$. Then: $S_2(W^{(n)}) \xrightarrow{w} \mathbb{B}$. in $C^{0,\alpha\text{-Höl}}$ for $\forall \alpha < \frac{1}{2}$

Rmk: Note $S_2(W^{(n)}) = \delta_{n^{-\frac{1}{2}}}(e^{\xi_1} \otimes \cdots e^{\xi_{[nt]}} \otimes e^{(nt-[nt])\xi_{[nt]+1}})$

$\in G^2(\mathbb{R}^d)$.

Thm (general type)

$(\tilde{\xi}_k)$ centered i.i.d $G^2(\mathbb{R}^d)$-valued r.v.'s. st. $\mathbb{E}(\|\xi_k\|^q) < \infty$. $\forall q, k \geq 1$. Set $W_0^{(n)} = 1$.

$W_t^{(n)} := \delta_{n^{-\frac{1}{2}}}(\xi_1 \otimes \cdots \xi_{[nt]})$. if $nt = [nt]$.

and take linear interpolation. Then:

$W^{(n)} \xrightarrow{w} \mathbb{B}$. in $C^{0,\alpha\text{-Höl}}([0,1], G^2(\mathbb{R}^d))$. $\forall \alpha < \frac{1}{2}$.