Conti. Semimart.

(1) <u>Definition:</u>

Note semimart. $S = M + V$. where M is c.l.m. V is FV process.

To lift S. we can lift M to $\mathbb{M}$ first. Then. just set

(lift of S)
$\mathbb{S} := T_V \mathbb{M}$. where we define:

$$T_V : C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d)) \times C^{q\text{-var}}([0,T], \mathbb{R}^d)$$

$$(\mathbb{X}, h)$$

$$\mapsto \text{plus}(S_{[p]}(\mathbb{X}, h)_t) \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$$

where $S_N(\mathbb{X}, h)$ is Young pairing of $(\mathbb{X}, h)$ of order N. (i.e. for proj.'s $p_N : G^N(\mathbb{R}^d \oplus \mathbb{R}^d) \to G^N(\mathbb{R}^d)$

and $p_N' : G_N(\mathbb{R}^d \oplus \mathbb{R}^{d'}) \to G_N(\mathbb{R}^d)$ $\Rightarrow$

$p_N(S_N(\mathbb{X}, h)) = S_N(\mathbb{X})$

$p_N'(S_N(\mathbb{X}, h)) = S_N(h)$)

And plus : $G^N(\mathbb{R}^d \oplus \mathbb{R}^d) \to G^N(\mathbb{R}^d)$

$$\exp(X \oplus g) \longmapsto \exp(X + g)$$

<u>Rmk</u>: i) If $\mathbb{X} = S_{[p]}(\tilde{h})$. where $\tilde{h}$ and h are FV processes. Then:

$$T_h(\mathbb{X}) = S_{[p]}(h + \tilde{h})$$

ii) We also have :

$$\|T_h(\mathbb{X})\|_{p\text{-var}, [0,T]} \leq C_{p,q}(\|\mathbb{X}\|_{p\text{-var}, [0,T]} + \|h\|_{q\text{-var}, [0,T]})$$

So : next, we only consider lift of M.

$M^c_{0,loc}([0,\infty), \mathbb{R}^d) \triangleq \{ M : [0,\infty) \to \mathbb{R}^d \mid M_0 = 0.$ M is. c.l.m $\}$.

<u>Pf</u>: For $M \in M^c_{0,loc}(\mathbb{R}^d)$.

i) $A: \Omega \times [0,\infty) \to \mathfrak{so}(d)$ area process. $A_t^{i,j} = \frac{1}{2}\left(\int_0^t M_r^i \, dM_r^j - M_r^j \, dM_r^i\right)$

ii) Enhanced c.l.m of M is $\mathbb{M} := e^{M+A} \in C([0,T], G^2(\mathbb{R}^d))$.

__Lemma.__ i) (ρ_s) is a.s. $\mathcal{F}$. right-conti. time changes. If M is const. on $[\rho_{t-}, \rho_t]$. Then: $M \circ \rho$ is c.l.m. and $\mathbb{M}$ is the lift.

ii) $\delta_c : G^N(\mathbb{R}^d) \to G^N(\mathbb{R}^d)$

$(x_0, x_1, \cdots x_N) \mapsto (x_0, cx_1, \cdots c^N x_N)$

If $(M, \mathbb{M})$ is pair of lift of c.l.m. Then so $(cM, \delta_c \mathbb{M})$ does.

__Cor.__ $(M, \mathbb{M})$ is pair of lift of c.l.m. $\Rightarrow (M^2, M^2)$ also is.

for τ stopping time

Pf: Set $\varphi_t = \tau \wedge t$.

(2) BDG. inequ.:

Def: $F: \mathbb{R}^+ \to \mathbb{R}^+$ is moderate if:

i) $F \in C(\mathbb{R}^+)$. increasing ii) $F(x)=0 \Leftrightarrow x=0$.

iii) $\exists\, q>1$. $\sup\limits_{x>0} \dfrac{F(qx)}{F(x)} < \infty$.

Rmk: i) e^x isn't moderate; x^k ✓.

ii) In fact iii) can work for $\forall q>1$: Since it's easy to see it works for $\{q^k\}_{k\geq 1}$. We use interpolation since F is monotone.

Lemma. i) F, G are moderate. Then: $F \circ G$ is moderate.

ii) F is moderate $\Rightarrow \exists\, c>0$. s.t. $\forall x, y>0$. $F(x+y) \leq c\,(F(x)+F(y))$

Thm. (General BDG inequality)

F is moderate. $M \in \mathcal{M}^c_{loc,0}(\mathbb{R}^d)$. τ is stopping time. Then. $\exists\, C = C(F, d)$.

$$C^{-1}\,\mathbb{E}(F(|\langle M\rangle_\tau|^{\frac{1}{2}})) \le \mathbb{E}(F(|M^*_\tau|)) \le C\,\mathbb{E}(F(|\langle M\rangle_\tau|^{\frac{1}{2}}))$$

RMK: i) $C(F,d)$ depends on growth para. of F. in fact. We can choose uniform $C(F,d)$ for $\{F_i\}$. if they have uniform growth const.

ii) For $M \in \mathcal{M}_{loc}$ only. we require: F is convex in addition!

Thm (For discrete mart.)

F moderate, convex ; $X: \mathbb{N} \to \mathbb{R}^d$ discrete local mart. $\langle X\rangle_n := \sum_{k=1}^{n} |\delta X_k|^2$. where $\delta X_k := X_k - X_{k-1}$. τ is stopping time. Then: $\exists\, C_{F,d}$ st. $\mathbb{E}(F(|X^*_\tau|)) \overset{C_{F,d}}{\sim} \mathbb{E}(F(\langle X\rangle_\tau^{\frac{1}{2}}))$

RMK: It doesn't work for x^p. $p < 1$ (even if X is mart.) as in conti. case.

Lemma. For F is moderate. $Y: \mathbb{N} \to \mathbb{R}^d$ is discrete mart. If $1 < q < p \leq 2$. or $p = q = 1$. Then $\exists\, c = c_{F,d}$. s.t.

$$\mathbb{E}(F(\|Y\|_{p\text{-var}})) \leq c\,\mathbb{E}(F((\sum_n |Y_{n+1} - Y_n|^2)^{\frac{1}{2}}))$$

where $\|Y\|_{p\text{-var}} := \sup_{(n_k)} \left(\sum |Y_{n_{k+1}} - Y_{n_k}|^p\right)^{\frac{1}{p}}$.

Thm. (BDG for enhanced mart.)

F is moderate. $\mathbb{M} \in \mathcal{M}^c_{loc}(G(\mathbb{R}^d))$ lift of c.l.m. M. Then $\exists\, c = c(F, d)$.

s.t. $C^{-1}\,\mathbb{E}(F(|\langle M \rangle_\tau|^{\frac{1}{2}})) \leq \mathbb{E}(F(\sup_{s,t \leq \tau} \|\mathbb{M}_{s,t}\|)) \leq C\,\mathbb{E}(F(|\langle M \rangle_\tau|^{\frac{1}{2}}))$

for stopping time τ.

Pf: WLOG set $\tau = \infty$.

i) Lower bdd: $\|\mathbb{M}_{s,t}\| \geq |M_{s,t}|$

ii) Upper bdd:

Note $\sup_{u,v} \|\mathbb{M}_{u,v}\| \leq C \sup_t \|\mathbb{M}_t\|$

with equi. of homo norm:

$\|\mathbb{M}_t\| \leq C(|M_t| + |A_t|^{\frac{1}{2}})$.

Using $F(x+y) \leq c(F(x)+F(y))$

We can just prove :

$$\mathbb{E}(F(\sup_{t\geq 0}|A_t|^{\frac{1}{2}})) \leq c\,\mathbb{E}(F(\langle M\rangle_\infty^{\frac{1}{2}})).$$

Note $|\langle A^{i,j}\rangle_t| \lesssim \sup|M_t|\cdot|\langle M\rangle_t|$.

with $F((\cdot)^{\frac{1}{2}})$ is moderate.

Apply BDG inequa. and use

moderate property of F

(3) Regularity :

prop. For $p>2$. $M \in \mathcal{M}^c_{0,loc}(G(\mathbb{R}^d))$

Then. $\forall T>0$. $\|M\|_{p\text{-var},[0,T]} < \infty$. a.s.

Pf: WLOG. set $M, \langle M\rangle$ bdd.

otherwise. set $\tau_n = \inf\{t \mid |M_t|>n \text{ or } |\langle M_t\rangle|>n\}$

$\mathbb{P}(\|M\|_{p\text{-var}} \neq \|M^{\tau_n}\|_{p\text{-var}}) \leq \mathbb{P}(\tau_n<T) \to 0$.

Next set time-change $\varphi(t) = \inf\{s \mid \langle m\rangle_s > t\}$. $X_t = m_{\varphi(t)}$

$\Rightarrow \mathbb{E}(\|X_{s,t}\|^{2q}) \overset{BDG}{\leq} C_q \mathbb{E}(|\langle m\rangle_{\varphi(t),\varphi(s)}|^q)$

$\leq C_q |t-s|^q$.

Apply Kolmogorov's Lemma. $\Rightarrow$

$\|X\|_{1/p\text{-Höl}}$ is L^q. $\forall q \geq 1$.

$\mathbb{P}(\|m\|_{p\text{-var}} > k) = \mathbb{P}(\|X\|_{p\text{-var}} > k)$

$\leq \dfrac{\mathbb{E}(\|X\|^2_{1/p\text{-Höl}})}{k^2} \xrightarrow{k\to\infty} 0$

<u>Lemma</u> (Chebyshev inequi.)

$\exists A$. for $\forall$ c.l.m M and $\forall \lambda > 0$.

$\mathbb{P}(\|M\|_{p\text{-var}, [0,\infty)} > \lambda) \leq A\, \mathbb{E}(|\langle m\rangle_\infty|)/\lambda^2$.

<u>Pf</u>: i) By scaling WLOG, set $\lambda = 1$.

By contradiction: $\forall k^2$. $\exists M^{(k)}$

$\in$ c.l.m. st. $k^2 \mathbb{E}(|\langle m^{(k)}\rangle_\infty|) < \mathbb{P}(\cdots > 1)$

Set $n_k := [\,1/\mathbb{P}(\|M^{(k)}\|_{p\text{-var}} > 1) + 1\,]$

and $(N^{(j)})_j$ is i.i.d. c.l.m of n_k

copy $M^{(k)}$, run on $\forall k$ on $(\Omega_j, \mathbb{P}_j)$

$\Rightarrow \sum_k \mathbb{P}_k(\|N^{(k)}\|_{p\text{-var}} > 1) = \infty$. and

$\sum_k \mathbb{E}^{\mathbb{P}_k}(|\langle N^{(k)}\rangle_\infty|) < \infty$

2) Define $\Omega = \otimes \Omega_k$. $\mathbb{P} = \otimes \mathbb{P}^k$

$\mathcal{F}_t = \bigotimes_1^{k-1} \mathcal{F}^i_\infty \bigotimes_{k+1}^{\infty} \mathcal{F}^i_0 \otimes \mathcal{F}^k_{g(k-t)}$. where

$g(t) = 1/t - 1$. Then:

$N_t := \sum_1^{k-1} N^{(i)}_\infty + N^{(k)}_{g(k-t)}$ $\quad k-1 \le t < k$.

concatenation of $N^{(k)}$. is a

conti. mart. on $(\Omega, (\mathcal{F}_t), \mathbb{P})$.

By 1): $\|N\|_{p\text{-var}}$ is a.s. infinite.

but $\mathbb{E}(|\langle N\rangle_\infty|) < \infty$.

Set $X_t = N_{1/(1-t)}$ on $[0,1)$. conti. mart.

So: $\|X_t\|_{p\text{-var}, [0,1]} = \|W\|_{p\text{-var}, [0,\infty)} = \infty$.

contradict with Thm. above!

<u>Thm</u> (BDG for p-var Norm)

F is moderate. $M \in \mathcal{M}^c_{0,loc}(G^2(\mathbb{R}^d))$,

Then. $\exists\, C = C(p, F, d)$. s.t.

$$C^{-1}\mathbb{E}(F(|\langle M\rangle_T|^{\frac{1}{2}})) \le \mathbb{E}(F(\|M\|_{p\text{-var}, [0,T]})) \le C\,\mathbb{E}(F(|\langle M\rangle_T|^{\frac{1}{2}}))$$

(4) <u>Approximations:</u>

Set M^D is piecewise linear approxi. of $M \in \mathcal{M}^c_{0,loc}$ in dissection D of $[0,T]$.

<u>THM.</u> F is moderate. $\exists\, C = C(p, F, d)$. s.t.

$$\mathbb{E}(F(\|S_2(M^D)\|_{p\text{-var}, [0,T]})) \le C\,\mathbb{E}(F(|\langle M\rangle_T|^{\frac{1}{2}}))$$

<u>THM.</u> If $\exists\, q > 1$. s.t. $|M|_{\infty, [0,T]} \in L^q$. Then:

$$\|S_2(M^D) - M\|_{p\text{-var}, [0,T]} \xrightarrow{L^q} 0 \;/\; \xrightarrow{pr} 0.$$