Gaussian Process

(1) **Motivations:**

For $X = (X_t^1 \cdots X_t^d)_{[1,T]}$ d-dim. centered Gaussian process with indept. components

It's characterize by its cov. func:

$R(s,t) = \text{diag}\{\mathbb{E}(X_s^1 X_t^1) \cdots \mathbb{E}(X_s^d X_t^d)\}$.

First, Note that:

$$\mathbb{E}((\int_1^t X_s^j \, dX_s^i)^2) = \int_{[1,t]^2} R_i(u,v) \, dR_j(u,v)$$

$$\lesssim \|R\|^2_{\ell\text{-var},[0,1]^2} . \quad \ell \in [1,2)$$

follows from Young inequi. and recall:

$$\|x\|_{p\text{-var},[1,T]^2} := \sup_{\substack{(t_i) \\ (t_j)} = [1,T]} \sum_{i,j} |x\begin{pmatrix} t_i, t_{i+1} \\ t_j, t_{j+1} \end{pmatrix}|^p$$

$$(\|R\|_{\ell\text{-var},[0,1]^2} \text{ mean}: \sup_{\substack{(t_i) \\ (t_j)}} \sum_{i,j} |\mathbb{E}((X_{t_{i+1}} - X_{t_i})(X^i_{t_{j+1}} - X^i_{t_j}))|$$

$\Rightarrow$ To to define: $\int X \otimes dX$. in L^2-sense. We require its cov. has finite ℓ-var. in 2D sense.

(2) <u>One-dim Gaussian</u>:

① <u>Def</u>: Fractional BM (fBM) β^H is centered Gaussian process, s.t. its cov. func. $R_{\beta^H}(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, $H \in (0,1)$.

<u>Rmk</u>: For $H > \frac{1}{2}$, then β^H has Hölder const. $> \frac{1}{2}$. And we notice $\beta^{\frac{1}{2}} = BM$.

<u>prop</u>. For $H \in (0, \frac{1}{2}]$. Then: $\exists\, C = C(H)$ s.t. $\| R_{\beta^H} \|_{\frac{1}{2H}\text{-var}, [s,t]^2} \leq C |t-s|^{\frac{1}{2H}}$. for $\forall\, s < t$. in $[0,1]$

Pf: Note $\mathbb{E}(\beta^n_{t_i,t_{i+1}}\beta^n_{t_j,t_{j+1}})<0$ if $i\neq j$.

$\Rightarrow \sum_{j\neq i} |\mathbb{E}(\beta^n_{t_i,t_{i+1}}\beta^n_{t_j,t_{j+1}})|^{\frac{1}{2n}}$ $\leq$ $(1/2n>1)$

$|\sum_{j\neq i}\mathbb{E}(\beta^n_{t_i,t_{i+1}}\beta^n_{t_j,t_{j+1}})|^{\frac{1}{2n}}$.

We have: $\sum_{i,j}|\mathbb{E}(\beta^n_{t_i,t_{i+1}}\beta^n_{t_j,t_{j+1}})|^{\frac{1}{2n}}$

$\lesssim_n \sum_i \mathbb{E}(|\beta^n_{t_i,t_{i+1}}|^2)^{\frac{1}{2n}} + |\mathbb{E}(\beta^n_{t_i,t_{i+1}}\beta^n_{s,t})|^{\frac{1}{2n}}$

$\lesssim_n |t-s|$. by calculation.

② Def: i) Cameron-Martin space $\mathcal{H} := \{ t\mapsto h_t = \mathbb{E}(Z X_t) \mid Z \in \overline{\operatorname{span}\{X_t\}_{[0,1]}}^{\|\cdot\|_{L^2}} \}$.

$\subset C([0,1],\mathbb{R}^1)$. associated with (X_t).

ii) $\langle h, h'\rangle_{\mathcal{H}} := \mathbb{E}(ZZ')$. if $h'_t = \mathbb{E}(Z'X_t)$.

Prop. (Embedding)

If R is ρ-variation in 2D sense for $\rho\in[1,\infty)$. Then: $\forall h\in\mathcal{H}$. $s<t$.

$\|h\|_{\rho\text{-var},[s,t]} \leq \langle h,h\rangle_{\mathcal{H}}^{\frac{1}{2}} \|R\|^{\frac{1}{2}}_{\rho\text{-var},[s,t]}$.

Rmk: $\mathcal{H} \underset{\text{conti}}{\hookrightarrow}$ finite ℓ-variation. $(\ell > 1)$

Pf: WLOG. Set $\langle h, h \rangle_{\mathcal{H}} = 1$.

$$\Big(\sum_j |h_{t_j, t_{j+1}}|^{\ell}\Big)^{\frac{1}{\ell}} = \sup_{|\beta|_{\ell'} = 1} \sum_j \beta_j h_{t_j, t_{j+1}}$$

$$= \sup \mathbb{E}\big(\sum \cdots\big) \overset{\text{Cauchy}}{\leq} \|R\|^{\frac{1}{2}}_{\ell\text{-var}, [s,t]}.$$

Cor. $\mathcal{H}^{BM}$ is CM space for $\mathbb{R}^1$-BM.

$\Rightarrow \mathcal{H}^{BM} = W_0^{1,2}([0,1], \mathbb{R}^1)$.

Rmk: $C_0^1([0,T], \mathbb{R}^1) \subset \mathcal{H}^{H}_{fBM}$ if X_t is fBM.

② **Pf:** X^D is linear piecewise approxi. for X

For $(s,t) \times (u,v) \subset (z_i, z_{i+1}) \times (\tilde{z}_i, \tilde{z}_{i+1})$

we set cov. func. of X^D and $X^{\tilde{D}}$:

$$R^{D,\tilde{D}}\begin{pmatrix} s,t \\ u,v \end{pmatrix} := \mathbb{E}\Big(\int_s^t dX_r^D \int_u^v dX_r^{\tilde{D}}\Big)$$

$$= \frac{t-s}{z_{i+1} - z_i} \cdot \frac{v-u}{\tilde{z}_{i+1} - \tilde{z}_i} R\begin{pmatrix} z_i, z_{i+1} \\ \tilde{z}_i, \tilde{z}_{i+1} \end{pmatrix}$$

Denote: $R^{\phi} = R^{0,0}$

Lemma. $\|R^{\phi}\|_{\substack{p\text{-var} \\ [u_1,v_1]\times \\ [u_2,v_2]}} \leq C_p \|R\|_{\substack{p\text{-var} \\ [u_1,v_1] \\ \times[u_2,v_2]}}$.

prop. (Hölder estimate)

Set $R_{(X,X^0)}\binom{s}{t} := \begin{pmatrix} \mathbb{E}(X_s X_t) & \mathbb{E}(X_s^0 X_t) \\ \mathbb{E}(X_s X_t^0) & \mathbb{E}(X_s^0 X_t^0) \end{pmatrix}$

If R is finite ϱ-variation

Then: $\|R\|^{\varrho}_{\substack{\varrho\text{-var} \\ [s,t]^2}} \leq k|t-s|$. $\forall s<t$

implies

$\Rightarrow \|R_{(X,X^0)}\|^{\varrho}_{\substack{\varrho\text{-var} \\ [s,t]^2}} \leq C k |t-s|$.

(3) Multidimensional Gaussian:

Consider $X = (X^1, \cdots X^d)$. centered $\mathbb{R}^d$-Gaussian process on $(E, \mathcal{H}, \mathbb{P})$ where $E = C([0,1], \mathbb{R}^d)$. and ($X_i$ are indept.) $\mathcal{H} \simeq \bigoplus_i^d \mathcal{H}^i$.

$\mathcal{H}^i$ is CM space of X^i.

① **prop.** If $X \in 1\text{-var}$. Set $\mathbb{X} = S_N(X)$ $\in G^N(\mathbb{R}^d)$. Then: $Z_n(\mathbb{X}_{s,t}) \in$ n^{th} Wiener chaos $\mathcal{H}_n$.

Rmk: Recall for $Z \in \mathcal{H}_n$, we have

$$\|Z\|_{L^2} \le \|Z\|_{L^q} \le (n+1)(q-1)^{\frac{n}{2}} \|Z\|_{L^2}.$$

prop. For $g \in G_N(\mathbb{R}^d)$, s.t. $\forall n$. $Z_n(g) \in \mathcal{H}_n$. n^{th} Wiener chaos. and $\exists$ w control on $[0,1]$. Satisfies:

$$\| Z_n(g_{s,t}) \|_{L^2} \le C\, w(s,t)^{n/2c}.$$

Then: i) $\| \|g_{s,t}\| \|_{L^q} \underset{\varepsilon, c, N}{\lesssim} w(s,t)^{\frac{1}{2c}}$. $\forall q \ge 1$.

ii) $\mathbb{E}\left(e^{\eta \|g\|^2_{p\text{-var},[0,1]}}\right) < \infty$. for some η. and $\forall p > 2c$.

Rmk: If $|w(s,t)| \le k|t-s|$. Then: $\|\cdot\|_{p\text{-var}}$ can be replaced by $\|\cdot\|_{1/p\text{-Höl}}$

② Uniform estimation:

Assum: X_t are finite variation and

$\exists \varrho \in [1,2)$, $\|R_X\|_{\varrho\text{-var},[0,1]^2} < \infty$. $R_X(s,t) \triangleq \mathbb{E}(X_s \otimes X_t)$

Prop. If i) R_X is ϱ-variation, $\varrho \in [1,2)$,

dominated by 2D control w.

ii) $\mathbb{X} = S_3(X)$.

Then: $\exists C = C(\varrho)$, for $\forall\, 1 \le n \le 3$.

$\| \pi_n(\mathbb{X}_{s,t}) \|_{L^2(\mathbb{P})} \le C\, w([s,t]^2)^{n/2\varrho}$.

Rmk: i) Apply the last prop. in ①

we have same results for $\mathbb{X}$.

ii) Consider $(X,Y) := (X_1, Y_1, \cdots, X_d, Y_d)$, 2d-dim Gaussian

process. s.t. (X_i, Y_i) indept.

The same result also holds.

③ Enhanced GPs:

Next, we weaken the assumptions in ②. Consider X a BV process but its R_X has ρ-variation $\rho \in [1,2)$.

Thm. If in addition, R_X is dominated by 2D control w, s.t. $w([0,1]^2) \le K$.

Then, $\exists$ unique conti. $G^2(\mathbb{R}^d)$-valued process $\mathbb{X}$, s.t.

i) $\mathbb{X}$ lifts X. i.e. $\pi_1(\mathbb{X}_t) = X_t - X_0$.

ii) $\exists\, C = C(\rho)$, for $\forall q \geq 1$. $\Rightarrow$

$$\| d(\mathbb{X}_s, \mathbb{X}_t) \|_{L^q} \le C\sqrt{q}\, w([s,t]^2)^{1/2\rho}$$

iii) $\forall p > 2\rho$. $\exists\, \eta = \eta(p, \rho, K)$, s.t.

$$\mathbb{E}\left(e^{\eta \|\mathbb{X}\|^2_{p\text{-var},[0,1]}}\right) < \infty.$$

iv) $\mathbb{X}$ is natural lift in sense that $S_3(X^n) \xrightarrow{a.s.} \mathbb{X}$. if X^n is a linear piecewise / mollifier approxi. of X ($\rho_\infty(X^n, X) \to 0$. a.s.)

Rmk: If $w(s,t) \leq K|t-s|$ in add. We can still replace $\|\cdot\|_{p\text{-var}}$ with $\|\cdot\|_{1/p\text{-Höl}}$

Def: We called $\mathbb{X}$ constructed above by enhanced Gaussian process / natural lift of X

Rmk: i) sample path of $\mathbb{X}$ is called Gaussian rough path since:

$\varrho \in [\frac{3}{2}, 2) \Rightarrow \mathbb{X}$ has p-variation for $p \in (2\varrho, 4)$. So $\in G^3(\mathbb{R}^d)$

$\varrho \in (1, \frac{3}{2}) \Rightarrow$ Proj. of $\mathbb{X}$ on $G^2(\mathbb{R}^d)$ has geometric p-path.

ii) The point iv) in Thm above guarantees uniqueness of $\mathbb{X}$.

e.g. $\tilde{\mathbb{X}}_t = (1, \pi_1(\mathbb{X}_t), \pi_2(\mathbb{X}_t)) \otimes e^{t[e_1, [e_1, e_2]]}$ also satisfies i) - iii) for $p \in [2,3)$.

iii) If X has a.s. finite $[1,2)$-variation. then $\mathbb{X}$ coincides with canonical lift in sense of iterated Young integral.

iv) BM can be lift to enhanced Gaussian process. It's identical with enhanced BM.

<u>prop.</u> Under conditions of Thm. above: we have: $\exists \eta = \eta(e, k) > 0$. st.

$$\sup_{0 \le s < t \le 1} \mathbb{E}\left(\exp\left(\eta\left(\frac{d(\mathbb{X}_s, \mathbb{X}_t)}{W(s,t)^{1/2e}}\right)^2\right)\right) < \infty.$$

④ Young Wiener Integral:

prop. If X is centred conti. $\mathbb{R}^1$-Gaussian process with R of finite e-variation. $h \in C^{q\text{-var}}([0,1], \mathbb{R}^1)$. s.t. $1/q + 1/e > 1$. Then: $\forall$ linear piece-wise / mollifier approxi. (X^n) to X. $\Rightarrow \int_0^t h\, dX^n$ converges in L^2 to limit denoted by $\int_0^t h\, dX$.

Besides, we have:

i) Young-Wiener isometry:

$$\mathbb{E}((\int_s^t h_u\, dX_u)^2) = \int_{[s,t]^2} h_u h_v\, dR(u,v)$$

ii) If $h(s)=0$, we have:

$$\mathbb{E}(|\int_s^t h_u\, dX_u|^2) \leq C_{e,q} \|h\|^2_{q\text{-var},[s,t]} \cdot \|R\|_{e\text{-var},[s,t]^2}$$

iii) $\int_0^t h\, dX$ is conti. p-variation for $p \geq q \wedge 2\varrho$.

<u>Pf:</u> i) Young-Wiener isometry holds if X is smooth. then use approxi.

ii) By Young 2D estimate:

$$\mathbb{E}(|\int_s^t h_u dX_u|^2) \lesssim \|h \otimes h\|_{q\text{-var}, [s,t]} \cdot \|R\|_{\varrho\text{-var}}$$

$$\lesssim \|h\|^2_{q\text{-var}} \|R\|_{\varrho\text{-var}}$$

So: $\sup_{t \in [0,1]} \mathbb{E}(|\int_0^t h\, dX^n - \int_0^t h\, dX^m|^2)$

$$\lesssim |h|^2_{q\text{-var}, [0,1]} \|R_{X^n - X^m}\|_{\varrho\text{-var}}$$

$$\lesssim \|R_{X^n - X^m}\|_\infty^{\varrho - \tilde{\varrho}/\varrho} \|R_{X^n - X^m}\|_{\tilde{\varrho}}^{\tilde{\varrho}/\varrho}$$

$\to 0$. So it's L^2-Cauchy

<u>Rmk:</u> Set $dR = \delta_{s=t}$, we cover the case of Ito's integral.

Cor. Under conditions above. If R_X is finite ℓ-variation and controlled by $2D$-control w. Then: $\int h \, dX$ is p-var. $\forall p > 2\ell$

Pf: Check Kolmogorov Lemma.

(4) Approxi.

Thm. For $X = (X^1, \cdots X^d)$ centered conti. GP with indept components and R of finite ℓ-variation. $\ell \in [1, 2)$. controlled by $2D$. cont. w. If $w([0,1]^2) \leq k$. Then. $\forall p \in (2\ell, 4)$.

$$\| \rho_{p\text{-var}, [0,1]} (\mathbb{X}, S_3(X^D)) \|_{L^2(P)} \leq C \sqrt{q} \max_i w([t_i, t_{i+1}]^2)^{\frac{\eta}{2}}.$$

for some $C = C(\ell, p, k, \theta)$, $\forall \eta \in (0, \frac{1}{2\ell} - \frac{1}{p})$. $\forall q \geq 1$. Where X^D is linear-piecewise approxi. of X.