Markov Process

(1) Motivation:

Next, we consider $X = X^a$ is diffusion with generator $L^a = \frac{1}{2} \sum_{i,j}^{d} \partial_i (a_{ij} \partial_j (\cdot))$

Rmk: If a is regular enough

$\Rightarrow X$ is semimart. and can be enhanced as before.

If $a(\cdot)$ is smooth. write $L^a =$

$\frac{1}{2} \sum_{i,j} a_{ij} \partial_i \partial_j + \frac{1}{2} \sum (\partial_i a_{ij}) \partial_j$

When $a = \sigma\sigma^T$. then X is solution of

$dX = \sigma(X) dB + b(X) dt$. $b = \sum \partial_i a_{ij} / 2$

And we define the lift in path-wise view as semimart.:

$t \mapsto A^{a;i,j}_{0,t} = \frac{1}{2} (\int_0^t X^{a;i}_{0,s} \, d X^{a;j}_{0,s} - \square)$.

$\Rightarrow (X^a_{0,t}, A^a_{0,t})$ is diffusion process on $\mathbb{R}^d \oplus so(d)$. Start at 0. With the generator $\mathcal{L}^a := \frac{1}{2}\sum_{i,j}^{d} U_i(a^{i,j} U_j(\cdot))$ where

$U_i|_x := \partial_i + \frac{1}{2}\left(\sum_{1\le j<i}^{d} x_j^1 \partial_{(j,i)} - x_i^1 \partial_{(j,i)}\right)$

define on $g^2(\mathbb{R}^d)$. $x = (x^1, x^2)$.

Rmk: i) $\langle \mathcal{L}^a f, g\rangle = \int_{g^2(\mathbb{R}^d)} \sum_{i,j}^{d} a^{ij} U_i f\, U_j g\, dm$

ii) path-area view and iterated-integral view are equi.:

$$(X, A) \underset{^2\log}{\overset{^2\exp}{\rightleftarrows}} \left(1, X, \int_0^\cdot X \otimes \circ dX\right)$$

By CBH formula it's consistent. $\leftarrow$ i.e. $g^2(\mathbb{R}^d)$ and $G^2(\mathbb{R}^d)$ are equi. in sense of Lie-isomor.:

Def: $(X, A) * (\tilde{X}, \tilde{A}) = \log\left(e^{(X,A)} \otimes e^{(\tilde{X},\tilde{A})}\right)$

$(X, A)^{-1} = (-X, -A)$. in $g^2(\mathbb{R}^d)$.

$$\Rightarrow {}^2\exp : (g^2(\mathbb{R}^d), *) \xrightarrow[\text{iso}]{\sim} (G^2(\mathbb{R}^d), \otimes)$$

We can also define metric:

$\|(x,A)\| = \| e^{(x,A)} \|_{G^2} \sim |x| + |A|^{\frac{1}{2}}$.

$d((x,A),(\tilde{x},\tilde{A})) = \|(x,A)^{-1} * (\tilde{x},\tilde{A})\|$.

Then we have $C^{\alpha\text{-Höl}}([0,T], g^N(\mathbb{R}^d))$.

iii) Besides, we can also define Lyon's lift for $C^{\alpha\text{-Höl}}([0,T], g^k(\mathbb{R}^d))$,

$S_N : C^{\alpha\text{-Höl}}([0,T], g^k(\mathbb{R}^d)) \to C^{\alpha\dots}(\dots g^N(\mathbb{R}^d))$

$= ({}^N\exp)^{-1} \circ S_N \circ ({}^k\exp)$.

where ${}^i\exp : g^i(\mathbb{R}^d) \xrightarrow{\sim} G^i(\mathbb{R}^d)$.

(2) Dirichlet form:

e.g. $g^2(\mathbb{R}^d) = \mathbb{R}^d \oplus so(d)$

Recall $g^N(\mathbb{R}^d) = \mathcal{L}^1(\mathbb{R}^d) \oplus \cdots \mathcal{L}^N(\mathbb{R}^d)$. $\{A \in \mathbb{R}^{d\times d}, A^T = -A\}$

① Def: i) Hypoelliptic gradient $\nabla^{hyp} := (u_1, \cdots u_d)^T$. u_i define in (1).

ii) $\Xi^{N,d}(\Lambda) = \{ a(\cdot) : g^N(\mathbb{R}^d) \to$ symmetric matrix, $|$ $a(\cdot)$ is measurable and

$\exists \Lambda > 0$ s.t. $\frac{1}{\Lambda}|\xi|^2 \le \xi \cdot a(\cdot)\xi \le \Lambda|\xi|^2$ for $\forall \xi \in \mathbb{R}^d$. }

iii) $I^a(f, g) := \nabla^{hyp} f \cdot a \nabla^{hyp} g = \sum_{i,j}^{d} a_{ij} u_i f \cdot u_j g$
for $f, g \in C_c^\infty(g^\wedge(\mathbb{R}^d), \mathbb{R}')$.

iv) $\mathcal{E}^a(f, g) := \int_{g^r(\mathbb{R}^d)} I^a(f, g)\, dm$, $dm(x)$
is Lebesgue measure on $g^\wedge(\mathbb{R}^d)$.

Denote: $\mathcal{E} = \mathcal{E}^I$, $I = I^I$.

prop. i) $\mathcal{E}^a$ is strongly local. (i.e. $f \in D(\mathcal{E})$ $\equiv$ const. on nbd of $\mathrm{supp}(g) \Rightarrow \mathcal{E}^a(f, g) = 0$)

ii) $\mathcal{E}^a$ can extend to a regular Dirichlet form, having $C_c^\infty(g, \mathbb{R}')$ as core. (i.e. C_c^∞ is dense in $(C_c, \|\cdot\|_{\mathcal{E}^a})$)

iii) $D(\mathcal{E}^a)$ is indept of choice of a. $\in \Xi(\Lambda)$. $D(\mathcal{E}^a) = \overline{C_c^\infty}^{\|\cdot\|_{W^{1,2}(g, dm)}}$

$\|f\|^2_{W^{1,2}(g, dm)} = \mathcal{E}(f, f) + \langle f, f \rangle_{L^2(g, dm)}$

Def: For $a \in \Xi^{r.d}(\Lambda)$. intrinsic distance associated with $\mathcal{E}^a$ is $d^a(x,y) =$

$\sup\{ f(x) - f(y) \mid f \in D(\mathcal{E}^a) \cap C_c(g^n(\mathbb{R}^d)).$

$\Gamma^a(f,f) \le 1 \}.$

prop. d^I is identical with Caratheodory - Carnot metric d on g.

Moreover: $d(x,y)/\Lambda^{\frac{1}{2}} \le d^a(x,y) \le \Lambda^{\frac{1}{2}} d(x,y)$.

prop. i) $B^a(x,r) \stackrel{\Delta}{=} \{ y \in g^n(\mathbb{R}^d) \mid d^a(x,y) < r \}$

$\Rightarrow \forall r > 0.\ x \in g^n(\mathbb{R}^d)$, we have:

$$m(B^a(x,2r)) \le 2^{C(\Lambda)} m(B^a(x,r))$$

ii) Weak Poincaré inequi: $\forall f \in D(\mathcal{E}^a)$.

$$\Rightarrow \int_{B^a(x,r)} |f - \bar{f}|^2 \, dm \le C r^2 \int_{B^a(x,2r)} \Gamma^a(f,f) \, dm$$

prop. (Harnack's inequi.)

$a \in \Xi^{r.d}(\Lambda)$. Consider u is nonnegative weak solution of $\partial_t u = \mathcal{L}^a u$. on Q^{cylinder}.

Then: i) $\exists\, C = C(\Delta)$. Q^+, Q^- separated cylinder of Q. s.t. $\sup_{Q^-} u \le C \sup_{Q^+} u$.

ii) $\exists\, \eta \in (0,1)$. $C = C(R, \Delta)$. s.t.

$$\sup_{\substack{(s,\eta)\\(s',\eta')}\in Q_1} |u(s,\eta) - u(s',\eta')| \le C \sup_{Q^*} |u| \left(\frac{|s-s'|^{1/2}}{r} + \frac{\hat{L}(\eta,\eta')}{r} \right)^{\eta}$$

Q_1, Q_2 are subcylinder of Q.

② <u>Construct X from $\mathcal{E}^\alpha$:</u>

Denote $(P_t^\alpha)_{t\ge 0}$ is L^2-semigroup of $\mathcal{L}^\alpha$.

$\exists\, p^\alpha(t,x,\eta) : (0,\infty) \times g^\alpha(\mathbb{R}^d) \times g^\alpha(\mathbb{R}^d)$. heat kernel st.

$P_t^\alpha f(\cdot) = \int f(\eta) p^\alpha(t,\cdot,\eta)\, d\omega(\eta)$. and it's

solution of $\partial_s u = \mathcal{L}^\alpha u$. $u(0,\cdot) = \delta_x$.

<u>Rmk</u>: $\mathcal{L}^\alpha$ is self-adjoint. So $p(t,\cdot,\cdot)$ is symmetric.

$\Rightarrow$ There exists symmetric diffusion process $X = X^{\alpha,x}$ associated to $\mathcal{L}^\alpha$.

given by $\mathbb{P}^{a,x}((X_{t_1}, \dots X_{t_n}) \in B) =$

$$\int_B p^a(t_1, x, y_1) \cdots p^a(t_n - t_{n-1}, y_{n-1}, y_n) \, dy_1 \cdots dy_n$$

Set $\mu^{a,x} = \mathbb{P}^{a,x} \circ X^{-1}$. law of

X on $C_x([0,\infty), \tilde{g}(\mathbb{R}^n))$. And X is

realization of coordinate of $C_x([0,\infty), \tilde{g})$.

<u>prop.</u> (heat kernel bdd)

Let $a \in \Xi^{r,d}(\Lambda)$. Then, $\forall t > 0$. $x, y \in \tilde{g}(\mathbb{R}^n)$.

we have: $\forall \varepsilon > 0$, $\exists C = C(\Lambda, \varepsilon)$. s.t.

$$C^{-1} \frac{1}{t^{c'}} e^{\frac{-C d^a(x,y)^2}{t}} \le p^a(t, x, y) \le \frac{C}{t^{c'}} e^{-C d^a(x,y)^2/4t}.$$

where $c' = c(\Lambda, \tilde{g}(\mathbb{R}^n))$.

<u>Cor.</u> $\forall \alpha \in (0, \frac{1}{2})$. $\exists C(T, \Lambda)$. we have:

$$\sup_{a \in \Xi(\Lambda)} \sup_{x \in \tilde{g}(\mathbb{R}^n)} \mathbb{E}^{a,x}\left(e^{C\eta \|X\|^2_{\alpha\text{-Höl}[0,1]}} \right) < \infty.$$

<u>Rmk</u>: There's a similar estimate for $\mathbb{X}_t$ with killing time.

<u>prop.</u> (Weak Scaling)

For $a \in \Xi(\Lambda)$. $a^r := a \circ \delta_r$. Then:

We have: $(\mathbb{X}_t^{a^r, x})_{t \geq 0} \overset{\text{law}}{\sim} (\delta_r \mathbb{X}^{a, \delta_{1/r}(x)}_{t/r^2})_{t \geq 0}$.

<u>Pf</u>: Note $(\mathbb{X}^a_{\lambda t})_{t \geq 0}$ corresp. the Dirichlet form $\lambda^2 \mathcal{E}^a$. and $(\delta_r \mathbb{X}^a_t)$ corresp. to $r^2 \mathcal{E}^{a^r}$.

(3) <u>Markovian Rough Path</u>:

① <u>Def</u>: i) For $N \geq 2$. $a \in \Xi^{N,d}(\Lambda)$. $\mathbb{X}_t$ constructed from $\mathcal{E}^a \in L^2$ is a.s. α-Hölder geometric rough path. for some $\alpha < \frac{1}{2}$. and is called Markovian RP.

ii) $N = 1$. $a \in \Xi^{1,d}(\Lambda)$. $a \circ \pi_1^{-1} \in \Xi^{2,d}(\Lambda)$.

Fix $x \in g^2(\mathbb{R}^d)$. The $g^2(\mathbb{R}^d)$ - valued

$\mathbb{X}^{a \circ Z_i^{-1}, x}$ constructed from $\mathcal{E}^{a \circ Z_i^{-1}} \in L^2$ is called natural lift of $X^{a \circ Z_i^{-1}, x}$.

<u>Rmk</u>: If $a \in \Xi^{1,d}(\Lambda)$ is regular enough s.t. X^a is semimart. Then enhance it in path-area view will have identical dist. as one in Def ii).

<u>Thm</u>. For $N \geq 2$, $\forall q \in (\frac{1}{N+1}, \frac{1}{N})$, $\mathbb{X} \in C^{q\text{-Höl}}([0,T], g^N(\mathbb{R}^d))$

② <u>Approxi.</u>

<u>prop</u>. If $a \in \Xi^{1,d}(\Lambda)$ is limit of smooth $a^n \in \Xi^{1,d}(\Lambda)$. Then: $\mathbb{X}^{a^n \circ Z_i^{-1}} \xrightarrow{w} \mathbb{X}^{a \circ Z_i^{-1}}$: Markovian rough path correspd Dirichlet form $\mathcal{E}^{a \circ Z_i^{-1}}$.

<u>Rmk</u>: It implies: $Z_i(\mathbb{X}^{a \circ Z_i^{-1}}) \sim X^a$.

Note $g^N(\mathbb{R}^d)$ is geodesic space. Set

x^D is geodesic approxi. for x

<u>prop.</u> $N \geq 2$. if $\mathbb{X} \in C^{\alpha\text{-Höl}}([0,1], G^N(\mathbb{R}^d))$, $\forall \alpha < \frac{1}{2}$

then: $\forall k \geq 2$, $d_{\alpha\text{-Höl}}(S_N(\pi_1(S_k(\mathbb{X}))^D), \mathbb{X})$

$\to 0 \quad (|D| \to 0)$

<u>Pf:</u> We only consider $k = 2$.

Since $S_N \circ \pi_1 \circ S_k \circ S_2 = S_N \circ \pi_1 \circ S_2$.

And Note: $(S_2(x))^D = S_2 \circ \pi_1(S_2(\mathbb{X}))^D$

$\Rightarrow S_N(S_2(x)^D) = S_N \circ \pi_1(S_2(\mathbb{X}))^D)$.

Note $\alpha \in (0, \frac{1}{2})$. We can use the continuity of Lyon's lift under $\|\cdot\|_{\alpha\text{-Höl}}$

<u>Rmk:</u> i) The approxi. requires priori know of path and area. i.e. $(\pi_1(\mathbb{X}), \pi_2(\mathbb{X}))$; if x^D is piecewise-linear. approxi. then $k = 1$ holds:

Note: ← segment line doesn't produce "area".

$d_{\alpha\text{-Höl}}(S_N(\pi_1(\mathbb{X})^D), \mathbb{X}) \xrightarrow[a.s.]{pr/L^p} 0$

ii) $k = 1$ doesn't hold: $(\vec{0}, \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix})$

$(N = 2, d = 2)$