RDEs

(1) Motivation:

Note for ODE $dy = V(y)dx = \sum_i^d V_i(y)dx^i$.

Lemma. $f(y_t) - f(y_s) = \sum_{k=1}^{n-1} \sum_{\substack{i_1 \cdots i_k \\ \in \{1,2\cdots d\}}} \int_{\Delta^k_{[s,t]}} V_{i_1} \cdots V_{i_k} f(y_s) \, dX^{i_1}_{r_1} \cdots dX^{i_k}_{r_k}$

$+ \sum_{\substack{i_1 \cdots i_n \\ \in \{1 \cdots d\}}} \int_{\Delta^n_{[s,t]}} V_{i_1} \cdots V_{i_n} f(y_{r_1}) \, dX^{i_1}_{r_1} \cdots dX^{i_n}_{r_n}$

where $V_i f := \sum_{k=1}^{e} V_i^k \partial_k f$. $V = (V_1 \cdots V_d)$

$V_i : (V_i^1 \cdots V_i^e) : \mathbb{R}^e \to \mathbb{R}^e$ $: \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$. $f \in C^n(\mathbb{R}^e, \mathbb{R}^1)$

Pf: $f(y)_{s,t} = \int_s^t Df(y)_r \cdot dy_r$

$= \int_s^t Vf(y)_r \, dX_r$

$= \int_s^t Vf(y)_s \, dX_r + \int_s^t (Vf(y)_r - Vf(y)_s) \, dX_r$

$= \cdots$ (iterated).

Def. For $g \in T^N(\mathbb{R}^d)$, $y \in \mathbb{R}^e$. N^{th}-Euler scheme is $\mathcal{E}_{(V)}(y, g) := \sum_{k=1}^{N} \sum_{\substack{i_1 \cdots i_k \\ \in \{1 \cdots d\}}} V_{i_1} \cdots V_{i_k} I(y) \, g^{k, i_1 \cdots i_k}$

I is identity on $\mathbb{R}^e$.

Note by argument of Lemma. We know:

$y_t \approx y_s + \mathcal{E}_{(V)}(y, S_N(x)_{s,t})$. for $|t-s|$ suff. small.

Rmk: The derivative we consider above is Frechét kind. Note V can be viewed: $y \mapsto [a=(a^1 \cdots a^d) \mapsto \sum_1^d V_i(y)a^i]$.

So: $V \in Lip^\gamma(\mathbb{R}^e) \iff V_i \in Lip^\gamma$. $\forall i$.

prop. (Estimate)

For $\gamma > 1$, $V \in Lip^{\gamma-1}(\mathbb{R}^e)$. $x \in C^{1\text{-var}}([s,t], \mathbb{R}^d)$. Then: $\exists C = C(\gamma)$. st.

$$|\pi_{(V)}(s, y_s; x) - \mathcal{E}_{(V)}(y_s, S_{[\gamma]}(x)_{s,t})|$$

$$\leq C (\|V\|_{Lip^{\gamma-1}} \int_s^t |dx_r|)^\gamma$$

pf: Apply the Lemma above with:

$$\left|\int_{\Delta^n_{[s,t]}} V_{i_1} \cdots V_{i_n} I(y_{r_1}) - V_{i_1} \cdots V_{i_n} I(y_s) dx^{i_1}_{r_1} \cdots\right|$$

$$\overset{\text{regular}}{\lesssim} \left|\int_{\Delta^n_{[r,t]}} dx^{i_1}_{r_1} \cdots dx^{i_n}_{r_n} \left(\int_s^t |dx_r|\right)^{\varepsilon(\gamma)}\right|$$

$$\lesssim \left(\int_s^t |dx_r|\right)^\gamma. \quad n = [\gamma].$$

Rmk: Note: $Z_{(V)}(0, y_0 ; X)_{0,T} - \mathcal{E}_{(V)}(y_0, S_n(X)_{0,T})$

$$= \sum_{\substack{i_1 \cdots i_n \\ \in \{1, \cdots, d\}}} \int_{0<r<T} (V_{i_1} \cdots I(y_r) - V_{i_1} \cdots I(y_0)) \, \underbrace{dX_r^{i_1}} \underbrace{\left(\int_{r<r_2 \cdots < r_n < T} dX_{r_2}^{i_2} \cdots dX_{r_n}^{i_n} \right)}$$

$d(X_{r,T}^{n; i_1 \cdots i_n}) =$ (check by def of $\frac{d}{dr}$)

$$\Rightarrow LHS = \int_0^T (V^n(y_r) - V^n(y_0)) \cdot d(X_{r,T}^n)$$

Lemma. (Davie's estimate)

For $\gamma > p \geq 1$. If: i) $V \in Lip^{\gamma - 1}$

ii) $X \in C^{1\text{-var}}([0,T], \mathbb{R}^d)$. $\mathbb{X} \triangleq S_{[p]}(X)$

iii) $y_0 \in \mathbb{R}^e$ is initial cond.

Then: $\exists\, C = C(p, \gamma)$. s.t. $\forall\, s < t \leq T$

$$\| Z_{(V)}(0, y_0 ; X) \|_{p\text{-var}, [s,t]} \leq C \left(\|V\|_{Lip^{\gamma-1}} \|\mathbb{X}\|_{p\text{-var}, [s,t]} \vee \|V\|^p_{Lip^{\gamma-1}} \|\mathbb{X}\|^p_{p\text{-var}} \right)$$

Cor. Under conditions above. if $X^s \in C^{1\text{-var}}([s,t], \mathbb{R}^d)$. st. $\int_s^t |dX^{s,t}| \leq K \|\mathbb{X}\|_{p\text{-var}}$

and $S_{[\gamma]}(X^{s,t})_{s,t} = S_{[\gamma]}(X)_{s,t}$

kind of stability

Thm: $|Z_{(v)}(s, y_s; X)_{s,t} - Z_{(v)}(\cdots; X^{s,t})_{s,t}|$

$\leq C_{\gamma,p} (k \|V\|_{Lip^{\gamma-1}} \|\mathbb{X}\|_{p\text{-}var,[s,t]})^{\gamma}$.

Rmk: i) Combine prop. above, we know

$\Sigma_{(v)}(s, y_s; S_{[\gamma]}(X^{s,t})_{s,t}) \xrightarrow{\text{approx.}}$

$Z_{(v)}(\cdots; X^{s,t})_{s,t} \xrightarrow{\text{approx.}} Z_v(\cdots X)_{s,t}$

ii) It gives a uniform estimate only depending on path regular.

(2) Solutions of RDEs:

Lemma (d_0/d_∞ estimate)

$d_0 \triangleq d_{0\text{-Hil}}$. On $C_0([0,1], G^N(\mathbb{R}^d))$

$\Rightarrow \exists C = C(N,d)$. s.t. $d_\infty(x,y)$

$\leq d_0(x,y) \leq C\, d_\infty(x,y) \vee d_\infty(x,y)^{\frac{1}{N}} (\|x\|_\infty + \|y\|_\infty)^{1-1/N}$

Recall $\forall\, x \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$. by prop. of geodesic span, we have $x_n \in C^{1\text{-var}}$.

$$\left[d_0(S_{[p]}(x_n), x) \to 0,\ \sup \| S_{[p]}(x_n) \| < \infty \right] (*)$$

By Lemma above, it also holds for d_∞ metric.

<u>Thm.</u> (Existence)

If i) $V \in Lip^{\gamma-1}$. $\gamma > p$ ii) y_0 is initial

iii) x_n is seq and $\mathbf{X}$ is weak geometric p-rough path. St. $(*)$ holds.

Then: At least along a subseq:

$\pi_{(V)}(0, y_0^n : x_n)$ converges to a limit $y \in C([0,T]; \mathbb{R}^e)$. under uniform top.

$$\text{St. } \|y\|_{\substack{p\text{-var}\\ [s,t]}} \le C_{p,\gamma} \left(\|V\|_{Lip^{\gamma-1}} \|\mathbf{X}\|_{\substack{p\text{-var}\\ [s,t]}} \vee \|V\|^p_{Lip^{\gamma-1}} \|\mathbf{X}\|^p_{p\text{-var}} \right)$$

Cor. Under the same conditions.

Cor. of the Davie's Lemma

also holds!

Pf: Apply Davie's Lemma on X_n.

$|Z_{(V)}(0, y_0^n; X_n)_{s,t}| \leq \cdots$

Combine with the conditions.

It follows from Ascoli's Thm.

Def: For $dy = V(y)\, d\mathbf{X}$. $y_0 \in \mathbb{R}^e$. $\mathbf{X} \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$. y is RDE solution driven by $\mathbf{X}$ along V start at y_0 if (*) holds. and $y_n \in Z_{(V)}(0, y_0; X_n)$. s.t. $y_n \xrightarrow{n} y$ on $[0,T]$.

Rmk: i) Davie's def: CAS approxi. of scheme' (Euler)

Davie defined y is RDE solution

if $\exists$ control $\tilde{w}$ and $\theta(\delta) = o(\delta)$ $(\delta \to 0)$

s.t. $|y_{s,t} - \mathcal{E}_{(V)}(y_s ; \mathbf{X}_{s,t})| \le \theta(\tilde{w}(s,t))$

Now we apply the cor. above. Set

$L\gamma \rfloor \ge p$. $\Rightarrow$ $\mathcal{S}_{[\gamma]}(\mathbf{X}) = \mathbf{X}$. $\tilde{w} = C \|\mathbf{X}\|^p_{p\text{-var}} \ldots)$

Rmk: The def led to:

$$y_{0,t} = \lim \sum y_{t_i^n, t_{i+1}^n} = \lim \sum \mathcal{E}_{(V)}(y_{t_i^n}, x_{t_i^n, t_{i+1}^n})$$

ii) Lyon's def:

Lyon defined the RDE solution as rough integral equation

Thm. (Local existence)

If we replace the condition: $Lip^{\gamma-1}$ by $Lip^{\gamma-1}_{loc}$ above. Then either there exists a global solution $y : [0,T] \to \mathbb{R}^e$ or $\exists\ z \in [0,T)$, s.t. y is solution on $[0,z)$. and $\lim_{t \uparrow z} |y(t)| = +\infty$.

Thm. (Uniqueness)

If i) $V^1, V^2 \in Lip^\gamma(\mathbb{R}^e)$. $\gamma > p \geq 1$

ii) w is fixed control.

iii) $\mathbb{X}^1, \mathbb{X}^2 \in C^{p\text{-}var}([0,T], G^{[p]}(\mathbb{R}^d))$. s.t.

$\|\mathbb{X}^i\|_{p,w} \leq 1$.

iv) $\|V^1\|_{Lip^\gamma} \vee \|V^2\|_{Lip^\gamma} \leq \nu < \infty$.

Then: $\exists$ unique solution $y^i = \pi_{(V^i)}(0, y_0^i; \mathbb{X}^i)$

s.t. $\rho_{p,w,[0,T]}(y^1, y^2) \hat{=} |y^1 - y^2|_{p,w,[0,T]} \leq C_{\gamma,p} e^{C\nu^p w(0,T)}$

$\cdot (\nu |y_0^1 - y_0^2| + |V^1 - V^2|_{Lip^{\gamma-1}} + \nu \rho_{p,w,[0,T]}(\mathbb{X}^1, \mathbb{X}^2))$

(3) Full RDE solution :

Next we extend the underlying space of y from $\mathbb{R}^e$ to $G^{[p]}(\mathbb{R}^e)$

Def : $\mathbb{X} \in C^{p\text{-}var}([0,T], G^{[p]}(\mathbb{R}^d))$. $y \in C([0,T],$
$G^{[p]}(\mathbb{R}^e))$ is full RDE solution along

(V_i) and start at $y_0 \in G^{[p]}(\mathbb{R}^e)$ if

$\exists (X^n) \in C^{1\text{-var}}([0,T], \mathbb{R}^d)$. st. $(*)$ hold

and $\exists y_n \in \pi_{(V)}(0, \pi_1(y_0); X^n)$. st.

$y_0 \otimes S_{[p]}(y_n) \longrightarrow y$ uniformly.

Thm. If i) $V \in \mathrm{Lip}^{\gamma-1}(\mathbb{R}^e)$. $\gamma > p$.

ii) $\mathbf{X}$ is weak geometric p-rough path

iii) $y_0 \in G^{[p]}(\mathbb{R}^e)$ initial condition

iv) y is full RDE solution of $(V, \mathbf{X})$.

Then: $u \mapsto z_u = y_u \in G^{[p]}(\mathbb{R}^e) \subset T^{[p]}(\mathbb{R}^e) \cong$

$\mathbb{R}^{1+e+\cdots+e^{[p]}}$ is solution of RDE:

$dz_u = W(z)\, d\mathbf{X}_u$. $W_i(z) = z \otimes V_i(\pi_1(z))$.

Pf: $dy_u = d(y_0 \otimes S_{[p]}(y)_{0,u})$

$\overset{\text{chain}}{\underset{\text{rule}}{=}} y_0 \otimes S_{[p]}(y)_{0,u} \otimes dy_{0,u}$

$\overset{\text{Def}}{=} y_u \otimes V(y_u)\, d\mathbf{X}_u$.

Rmk: The existence and uniqueness Thm are identical in case of (2). And the THMs can be extended to $X \in C^{\psi_{p,1}\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$, where $\psi_{p,1} = t^p / \ln\ln(\ln\ln(1/t))$.

$\Rightarrow$ It can also be applied in BM which is $\psi_{2,1}$-var but not 2-var.

Note when $V = (V_i): z \mapsto (A_i z + b_i)$ is linear vector field. Then bdd condition won't hold. The THMs above can only assert local existence.

Thm. If i) $V_i(z) = A_i z + b_i$. $\upsilon \geq \max(|A_i| + |b_i|)$

ii) $x \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$. $y_0 \in G^{[p]}(\mathbb{R}^e)$ is initial condition.

Then: $\exists$ unique full RDE solution: $Z_{(V)}(0, y_0; x)$ on $[0,T)$ and it satisfies

$\| Z(v)(0, y_0; X)_{s,t} \| \leq C_p (1 + |y_0|) \vee \|X\|_{p\text{-var}}$.

<u>Rmk</u>: The estimate can't be improve! $\quad e^{C v^p \|X\|^p_{p\text{-var}}}$

(4) Integration along Rough path:

Note that we just define RDE solution as limit of ODE solutions.

Next, we also def rough integration as limit of R-S integrals.

<u>Def</u>: $\mathbb{X} \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$. $\varphi = (\varphi_i)_1^d : \mathbb{R}^d \longrightarrow \mathbb{R}^e$. $Y \in C([0,T], G^{[p]}(\mathbb{R}^e))$

is rough path integral of φ along $\mathbb{X}$ if $\exists (X_n) \subset C^{1\text{-var}}([0,T], \mathbb{R}^d)$. s.t.

i) $X_0^n = \pi_1(\mathbb{X})$, $\forall n$. ii) $d_{0,[0,T]}(S_{[p]}(X^n), \mathbb{X}) \to 0$

iii) $\sup_n \| S_{[p]}(X^n) \|_{p\text{-var},[0,T]} < \infty$

iv) $\lim_{n} S_{[p]}(\int_0^{\cdot} \varphi(\hat{x}_n) d\hat{x}_n, y) \to 0$

Rmk: $\int \varphi(x) d\mathbf{X}$ denotes set of such y.

Thm. If i) $(\varphi_j)_1^d \subset Lip^{\gamma-1}(\mathbb{R}^d, \mathbb{R}^e)$, $\gamma > p \geq 1$

ii) $\mathbf{X} \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$.

Then: $\forall s < t \in [0,T]$, there exists unique rough path integral of φ along $\mathbf{X}$, which is geometric rough path. And

$\| \int \varphi(x) d\mathbf{X} \|_{p\text{-var}, [s,t]} \leq C_{p,\gamma} \|\varphi\|_{Lip^{\gamma-1}} (\|\mathbf{X}\|_{p\text{-var}} \vee \|\mathbf{X}\|^p_{p\text{-var},[s,t]})$

Rmk: $\int \varphi(x) d\mathbf{X}$ also have continuity property if $\exists$ fixed control w.

s.t. $\max_{i=1,2} \{ \|\varphi^i\|_{Lip^{\gamma-1}}, \|\mathbf{X}^i\|_{p,w,[0,T]} \} \leq R < \infty$.

(5) RDEs with Drift:

Next, we consider $V(y) d\ddot{\mathbf{X}}$ in ODE is to model state-dependent perturbation:

Then the RDE has form:

$dy = V(y)\, d\mathbf{X} + W(y)\, dt$. $W(y)dt$ is drift.

<u>Rmk</u>: We can replace V by $\tilde{V} = (V, W)$

and replace $\mathbf{X}$ by $\tilde{\mathbf{X}} = S_{[p]}(\mathbf{X}, t)$

$\Rightarrow dy = \tilde{V}(y)\, d\tilde{\mathbf{X}}$.

(*)
Next, we consider more general form.

$W = (W_i)_1^{e}$. $h \in C^{q\text{-var}}([0,T], G^{[q]}(\mathbb{R}^{e}))$. To let

the Young pairing $S_{[p]}(\mathbf{X}, h)$ well-def:

Assume $1/p + 1/q > 1$.

<u>Rmk</u>: To get the datum: set $q = 1$. $h_t = t$.

<u>Def</u>: Under the setting of (*). we say

$y \in C([0,T], \mathbb{R}^e)$ is RDE solution of

$dy = V(y)\, d\mathbf{X} + W(y)\, dh$. Start at y_0

if $\exists (x^n, h^n) \subset C^{1\text{-var}}([0,T], \mathbb{R}^d) \times C^{1\text{-var}}(\cdots, \mathbb{R}^{d'})$.

s.t. $\sup_n \|S_{[p]}(x^n)\|_{p\text{-var}, [0,T]} + \|S_{[q]}(h^n)\|_{q\text{-var}, [0,T]} < \infty$

$\lim_{n\to\infty} d_{0,[0,T]}(S_{[p]}(x^n), \mathbf{X}) = \lim_{n\to\infty} d_{0,[0,T]}(S_{[q]}(h^n), \mathbf{h})$

$= 0$ and $y_n \in \pi_{(V,W)}(0, y_0; (x^n, h^n))$. s.t.

$y_n \xrightarrow{u} y$ on $[0,T]$. $(n\to\infty)$

<u>Rmk</u>: As before. to define full RDE

solution $\mathbf{y} \in C([0,T], G^{[p\vee q]}(\mathbb{R}^e))$.

we require $\mathbf{y}_0 \otimes S_{[p\vee q]}(y^n) \to \mathbf{y}$

in $[0,T]$. $\mathbf{y}_0 \in G^{[p\vee q]}(\mathbb{R}^d)$ starting pt.

Theorem 12.6 *Assume that, $p, q, \gamma, \beta \in [1, \infty)$ are such that*

$$1/p + 1/q > 1 \tag{12.2}$$

$$\gamma > p \text{ and } \beta > q \tag{12.3}$$

$$\frac{\gamma - 1}{q} + \frac{1}{p} > 1 \text{ and } \frac{1}{q} + \frac{\beta - 1}{p} > 1; \tag{12.4}$$

(i) $V = (V_i)_{1 \le i \le d}$ is a collection of vector fields in $\mathrm{Lip}^{\gamma-1}(\mathbb{R}^e)$;
(i bis) $W = (W_i)_{1 \le i \le d'}$ is a collection of vector fields in $\mathrm{Lip}^{\beta-1}(\mathbb{R}^e)$;
(ii) (x_n) is a sequence in $C^{1\text{-var}}([0,T], \mathbb{R}^d)$, and $\mathbf{x}$ is a weak geometric p-rough path such that

$$\lim_{n\to\infty} d_{0,[0,T]}(S_{[p]}(x_n), \mathbf{x}) \text{ and } \sup_n \|S_{[p]}(x_n)\|_{p\text{-var};[0,T]} < \infty;$$

(ii bis) (h_n) is a sequence in $C^{1\text{-var}}([0,T], \mathbb{R}^{d'})$, and $\mathbf{h}$ is a weak geometric q-rough path such that

$$\lim_{n\to\infty} d_{0,[0,T]}(S_{[q]}(h_n), \mathbf{h}) \text{ and } \sup_n \|S_{[q]}(h_n)\|_{q\text{-var};[0,T]} < \infty; .$$

(iii) $\mathbf{y}_0^n \in G^{[\max(p,q)]}(\mathbb{R}^e)$ *is a sequence converging to some* $\mathbf{y}_0$*;*
(iv) ω *is the control defined by*

$$\omega(s,t) = \left(|V|_{\mathrm{Lip}^{\gamma-1}} \|\mathbf{x}\|_{p\text{-}var;[s,t]} \right)^p + \left(|W|_{\mathrm{Lip}^{\beta-1}} \|\mathbf{h}\|_{q\text{-}var;[s,t]} \right)^q .$$

that ←
full RDE
solution

Then, at least along a subsequence, $\underline{\mathbf{y}_0^n \otimes S_{[\max(p,q)]} \left(\pi_{(V,W)} \left(0, \pi_1(\mathbf{y}_0^n); (x_n, h_n)\right)\right)}$ *converges in uniform topology, and there exists a constant* C_1 *depending on* p*,* q*,* γ*, and* β *such that for any limit point* $\mathbf{y}$*, and all* $s < t$ *in* $[0,T]$*,*

$$\|\mathbf{y}\|_{\max(p,q)\text{-}var;[s,t]} \leq C_1 \left(\omega(s,t)^{1/\max(p,q)} \vee \omega(s,t) \right).$$

<u>Rmk</u>: For the uniqueness, we replace $Lip^{\gamma-1}$, $Lip^{\beta-1}$ by Lip^{γ}, Lip^{β} as before. And require:

$\|X\|_{p\text{-}var}$, $\|h\|_{q\text{-}var}$, $\|V\|_{Lip^\gamma}$, $\|W\|_{Lip^\beta}$ are all bdd.