Conti. Martingales.

(1) Definitions:

Consider prob space $(\Omega, \mathcal{F}, P)$.

Def: i) A filtration on $(\Omega, \mathcal{F}, P)$ is collection $(\mathcal{F}_t)_{0\le t\le\infty}$ sub σ-fields of $\mathcal{F}$. st. $\mathcal{F}_s \subset \mathcal{F}_t$. $\forall\, s\le t\le\infty$.

ii) For random process $(X_t)_{t\ge0}$. its canonical filtration is $\mathcal{F}^X_t =: \sigma(X_s,\ 0\le s\le t)$. $\mathcal{F}^X_\infty =: \sigma(X_s,\ s\ge0)$

iii) Denote $\mathcal{F}_{t+} = \bigcap_{s>t}\mathcal{F}_s$. $\mathcal{F}_{\infty+} = \mathcal{F}_\infty$. $(\mathcal{F}_{t+})$ is filtration as well. We say filtration $(\mathcal{F}_t)$. is right-conti if $\mathcal{F}_{t+} = \mathcal{F}_t$. $\forall\, t\ge0$.

iv) $(\mathcal{F}_t)$ is complete if $\mathcal{N}\subset\mathcal{F}_0$. where $\mathcal{N} = \{A\in\mathcal{F}_\infty \mid P(A)=0\}$.

Rmk: $(\mathcal{F}_t)$ can be completed by: $\mathcal{F}'_t = \mathcal{F}_t \vee \sigma(\mathcal{N})$.

Def: i) Process $X=(X_t)_{t\ge0}$ takes values in measurable space $(E,\mathcal{E})$. is measurable if: $\Omega\times\mathbb{R}^+ \xrightarrow{X} E$, $(w,t)\mapsto X_t(w)$ is $\mathcal{F}\otimes\mathcal{B}_{\mathbb{R}^+}$ - measurable.

ii) Process $X=(X_t)_{t\ge0}$ is adapted if $X_t\in\mathcal{F}_t$. $\forall t$.

iii) $X=(X_t)$ is progressive if $\forall\, t\ge0$: $\Omega\times[0,t]\longrightarrow E$, $(w,s)\longmapsto X_s(w)$. is measurable for $\mathcal{F}_t\otimes\mathcal{B}_{[0,t]}$.

Rmk: Progressive $\Longrightarrow$ adapted + measurable.

Prop. (X_t) takes values in metric space (E, d), equipped with Borel σ-field. Then:

X is adapted, right-conti for every $w \in \Omega$

$\Rightarrow$ X is progressive.

Pf: Fix $t > 0$. for $s \in [0,t]$. Def: $X_s^n = X_{kt/n}$ if $\frac{(k-1)t}{n} \le s < \frac{kt}{n}$

and $X_t^n = X_t \Rightarrow X_s(w) = \lim_n X_s^n(w)$. by right-conti.

Besides. $\{X_s^n \in A\} = (\{X_t \in A\} \times \{t\}) \cup (\bigcup_k \{X_{\frac{kt}{n}} \in A\} \times [\frac{(k-1)t}{n}, \frac{kt}{n}))$

$\in \mathcal{F}_t \otimes \mathcal{B}_{[0,t]}$.

So limit of $X_s^n(w) = X_s(w)$ is $\mathcal{F}_t \otimes \mathcal{B}_{[0,t]}$-measurable.

Def: i) Progressive σ-field is collection $\mathcal{P}$ of all sets $A \in \mathcal{F} \otimes \mathcal{B}_{\mathbb{R}^+}$ s.t. $I_A(w,t)$ is progressive.

ii) Predictable σ-field is σ-field generated by predictable rectangles: $F \times (s,t]$, $F \in \mathcal{F}_s$.

Lemma. Predictable σ-field is equi. with:

i) σ-field generated by all conti adapted process.

ii) σ-field generated by all right-conti adapted process.

iii) σ-field generated by all adapted càglàd process.

Rmk: We say predictable process is measurable w.r.t predictable σ-algebra.

Note: Predictable $\Rightarrow$ Progressive.

But converse may not hold in some case.

(2) Stopping Times:

① Definitions:

Def: i) r.v. $T: \Omega \to \overline{\mathbb{R}}_+$ is stopping time w.r.t $(\mathcal{F}_t)$

if $\{T \le t\} \in \mathcal{F}_t$. $\forall t \ge 0$.

ii) $\mathcal{F}_T = \{A \in \mathcal{F}_\infty \mid \forall t \ge 0.\ A \cap \{T \le t\} \in \mathcal{F}_t\}$.

prop. i) r.v. $T: \Omega \to \overline{\mathbb{R}}_+$ is stopping time of $(\mathcal{F}_{t+})$

$\iff \{T < t\} \in \mathcal{F}_t$. $\iff T \wedge t \in \mathcal{F}_t$. $\forall t > 0$.

ii) T is stopping time w.r.t $(\mathcal{F}_{t+})$. Then:

$\mathcal{F}_{T+} = \{A \in \mathcal{F}_\infty \mid \forall t > 0.\ A \cap \{T < t\} \in \mathcal{F}_t\}$.

Pf: i) $(\Rightarrow)$. $\{T < t\} = \bigcup_{\substack{q < t \\ q \in \mathbb{Q}_+}} \{T \le q\} \in \mathcal{F}_t$. by $\{T \le q\} \in \mathcal{F}_{q+} \subset \mathcal{F}_t$

$(\Leftarrow)$ $\forall s > t \ge 0$. $\{T \le t\} = \bigcap_{\substack{t < q < s \\ q \in \mathbb{Q}_+}} \{T < q\} \in \mathcal{F}_s$

$\Rightarrow \{T \le t\} \in \mathcal{F}_{t+}$.

For the last part: $T \wedge t \in \mathcal{F}_t \iff \{T \le s\} \in \mathcal{F}_t$. $s < t$.

ii) $(\Rightarrow)$ For $A \in \mathcal{F}_{T+}$. Then $A \cap \{T \le t\} \in \mathcal{F}_{t+}$

So $A \cap \{T < t\} = \bigcup_{q \in \mathbb{Q}^+ < t} (A \cap \{T \le q\}) \in \mathcal{F}_t$

$(\Leftarrow)$ Similar to part i) $(\Leftarrow)$.

Rmk: T is stopping time w.r.t $(\mathcal{F}_t) \Rightarrow$ w.r.t $(\mathcal{F}_{t+})$

but converse isn't true!

② Properties:

i) $\forall$ stopping time T. $\mathcal{F}_T \subset \mathcal{F}_{T+}$. If $\mathcal{F}_t$ is right-conti

Then: $\mathcal{F}_T = \mathcal{F}_{T+}$.

ii) $T \in \mathcal{F}_T$. for T is stopping time.

iii) T is stopping time. $A \in \mathcal{F}_\infty$. Set:

$$T^A(w) = \begin{cases} T(w). & w \in A. \\ +\infty. & w \notin A. \end{cases}$$ Then: $A \in \mathcal{F}_T \iff$

T^A is stopping time w.r.t. $(\mathcal{F}_t)$.

iv) $S.T.$ stopping times. $S \le T$. Then: $\mathcal{F}_S \subset \mathcal{F}_T$.

and $\mathcal{F}_{S+} \subset \mathcal{F}_{T+}$.

v) $S.T$ stopping times. Then: $S \wedge T$. $S \vee T$ are stopping times. Moreover. $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

$\{S \le T\}$. $\{S = T\} \in \mathcal{F}_{S \wedge T}$.

vi) (S_n) stopping times. $\uparrow$. Then $S_n \uparrow S$ is also a stopping time.

vii) (S_n) stopping times $\downarrow$. Then $S_n \downarrow S$ is stopping time w.r.t $(\mathcal{F}_{t+})$. $\mathcal{F}_{S+} = \bigcap_n \mathcal{F}_{S_n+}$.

viii) (S_n) stopping times $\downarrow$. stationary (i.e. $\forall w$. $\exists N(w)$. st. $S_n(w) = S(w)$. $\forall n \ge N(w)$) Then $S_n \downarrow S$ is also stopping time. $\mathcal{F}_S = \bigcap \mathcal{F}_{S_n}$.

ix) $w \mapsto Y(w) \in (E, \mathcal{E})$. define on set $\{T < \infty\}$.

Then: $Y \in \mathcal{F}_T \iff \forall t \ge 0$. $Y|_{\{T \le t\}} \in \mathcal{F}_t$.

<u>Pf:</u> v) $\{S \le T\} \cap \{T \le t\} = \{S \le t\} \cap \{T \le t\} \cap \{S \wedge t \le T \wedge t\}$.

$\{S \le T\} \cap \{S \le t\} = \{S \le t\} \cap \{S \wedge t \le T \wedge t\}$.

vii) $\{S < t\} = \bigcup_n \{S_n < t\}. \in \mathcal{F}_t$.

viii) In stationary case. we truly have:

$$\{S \le t\} = \bigcup \{S_n \le t\}.$$

ix) $Y \in \mathcal{F}_T \iff \{Y \in A\} \in \mathcal{F}_T.\ \forall A \in \mathcal{E}.$

$\iff \{Y \in A\} \cap \{T \le t\} \in \mathcal{F}_t.$

Cor. i) $S.\ T$ stopping times. Then so $S+T$ is.

ii) (T_n) stopping times. Then. $\sup T_n$. $\inf T_n$.

$\overline{\lim}\, T_n$. $\underline{\lim}\, T_n$ are stopping times.

Pf: It follows from: v).

Thm. $(X_t)_{t \ge 0}$ progressive process with values in $(E, \mathcal{E})$.

T is stopping time. Then: $w \longmapsto X_{T(w)}(w)$ defined

on $\{T < \infty\}$ is $\mathcal{F}_T$-measurable.

Pf: By property ix). Restrict on $\{T \le t\}$.

It's composition of: $w \in \{T \le t\} \longmapsto (w, T(w) \wedge t)$

and $(w, s) \in \Omega \times [0, t] \longmapsto X_s(w)$. both measurable.

prop. T is stopping time. $S \in \mathcal{F}_T$. takes value in $[0, \infty]$.

st. $S \ge T$. Then S is stopping time.

pf: $\{S \le t\} = \{S \le t\} \cap \{T \le t\} \in \mathcal{F}_t$.

Cor. $T_n = \sum \frac{k+1}{2^n} I_{\{k/2^n < T \le (k+1)/2^n\}} + \infty \cdot I_{\{T = \infty\}}$

is seq of stopping times $\downarrow T$.

prop. (X_t) adapted. takes value in $(E, \mathcal{E})$. metric space.

i) If sample paths of X are right-conti. and

O is open in E. Then $T_O = \inf\{t>0 \mid X_t \in O\}$

is stopping time w.r.t. $(\mathcal{F}_{t+})$.

ii) If sample paths of X are conti. F is closed

in E. Then $T_F = \inf\{t>0 \mid X_t \in F\}$ is stopping time

<u>Pf</u>: i) $\{T_O < t\} = \bigcup_{s\in[0,t)\cap \mathbb{Q}} \{X_s \in O\} \in \mathcal{F}_t$

ii) $\{T_F \le t\} = \{\inf_{s\in[0,t]\cap \mathbb{Q}} d(X_s, F) = 0\} \in \mathcal{F}_t$.

<u>Rmk</u>: ii) can be extended to X is right-conti.

(but not in this method. Since $f(s) = d(X_s(\omega), F)$

is just right-conti. if X just right-conti)

<u>Thm</u>. (Debut's)

For $X = (X_t)$ is progressive. If $K \in \mathcal{E}$ measurable set.

Then: $T_K = \inf\{t>0 \mid X_t \in K\}$ is stopping time.

(3) <u>Martingales</u>:

① <u>Def</u>: On $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. An adapted real valued

process (X_t) is martingale if:

$X_t \in L^1$. $\forall t \ge 0$ and $E(X_t \mid \mathcal{F}_s) = X_s$. $\forall 0 \le s < t$.

<u>e.g.</u> i) $Z_t \in L^1$. $\forall t \ge 0$. $\tilde{Z}_t = Z_t - E(Z_t)$ is mart.

ii) $Z_t \in L^2$. $\forall t \ge 0$. $Y_t = \tilde{Z}_t^2 - E(\tilde{Z}_t^2)$ is mart.

iii) For some $\theta \in \mathbb{R}^1$. $E(e^{\theta Z_t}) < \infty$. $\forall t \ge 0$. Then:

$X_t = e^{\theta Z_t} / E(e^{\theta Z_t})$ is mart.

i). ii). iii) holds if (Z_t) is adapted and has indept. increment. (i.e. $Z_t - Z_s$ indept of $\mathcal{F}_s$)

<u>prop.</u> (X_t) adapted. $f: \mathbb{R} \to \mathbb{R}^+$. convex. $E(f(X_t)) < \infty$. $\forall t$.

i) If (X_t) is mart. Then $f(X_t)$ is submart.

ii) If (X_t) is submart. $f \nearrow$. Then $f(X_t)$ is submart.

<u>Cor.</u> (X_t) mart $\Rightarrow |X_t|^p$. X_t^+ submart.

<u>prop.</u> (X_t) is submart. Then: $\forall t > 0$. $\sup\limits_{0 \le s \le t} E(|X_s|) < \infty$.

<u>pf.</u> (X_t^+) is submart. $\Rightarrow E(X_s^+) \le E(X_t^+)$. $\forall 0 \le s \le t$.

Combined with $E(X_s) \geq E(X_0)$. by X is submart.

$\Rightarrow \sup\limits_{0 \le s \le t} E(|X_s|) \le 2E(X_t^+) - E(X_0)$

<u>prop.</u> (Square integrable mart.)

$(M_t) \subset L^2$. mart. If $s = t_0 < t_1 \cdots < t_p = t$ is subdivision of $[s,t]$. Then: $E(\sum\limits_1^p (M_{t_i} - M_{t_{i-1}})^2 | \mathcal{F}_s) = E(M_t^2 - M_s^2 | \mathcal{F}_s)$

$= E((M_t - M_s)^2 | \mathcal{F}_s)$.

<u>pf.</u> $E((M_{t_i} - M_{t_{i-1}})^2 | \mathcal{F}_s) = E(E(M_{t_i} - M_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}) | \mathcal{F}_s)$

$= E(M_{t_i}^2 - M_{t_{i-1}}^2 | \mathcal{F}_s)$.

Thm. (Inequilities)

i) For (X_t) right-conti. supermart. Then $\forall t > 0$, $\lambda > 0$.

$\lambda P(\sup\limits_{s \le t} X_s^- \geq \lambda) \le E(X_t I_{\{\sup X_s^- \geq \lambda\}}) \le E(X_t^- I_{\{\sigma\}})$

$\lambda P(\sup\limits_{s \le t} |X_s| \geq \lambda) \le E|X_0| + 2E|X_t|$.

ii) (X_t) is right-conti mart. Then. $\forall t>0$. $p>1$.

$$E(\sup_{s\le t}|X_s|^p) \le \left(\frac{p}{p-1}\right)^p E(|X_t|^p).$$

Pf: Consider $D_m = \{kt/2^m \mid 0\le k\le 2^m\} =: (t_k^m)$

$D_m \uparrow D = \cup D_m$ is dense in $[0,t]$.

Set $Y_k = X_{t^m_{k\wedge 2^m}}$ is discrete supermart. w.r.t. $\mathcal{F}_{t^m_{k\wedge 2^m}}$.

$\Rightarrow$ In discrete case: $2E|X_t| + 2E|X_0| \ge \lambda P(\max_{D_m}|X_s|\ge\lambda)$

RHS $\uparrow$ $P(\sup_{D\cap[0,t]}|X_s|\ge\lambda)$. Note: $\sup_{D\cap[0,t]}|X_s| = \sup_{[0,t]}|X_s|$

Other two ineq. is proved similarly by MCT.

Rmk: For general case: X is supermart. $\forall D$ is countable dense set. We have: $\forall\lambda, t>0$.

$$P(\sup_{D\cap[0,t]}|X_s|>\lambda) \le \frac{1}{\lambda}(2E|X_t| + E|X_0|)$$

Let $\lambda\to\infty \Rightarrow \sup_{s\in D\cap[0,t]}|X_s| < \infty$. a.s.

③ Upcrossing number and limit:

Def: i) $f: \mathbb{R}_+ \to \mathbb{R}^1$ is cadlag if f is right-conti and left-limits exist for every point.

ii) The upcrossing number of $f: I\subset\mathbb{R}^+\to\mathbb{R}^1$ on $[a,b]$ is denoted by $M^f_{ab}(I)$ which is the max integer $k\ge1$ st. $\exists$ seq. $s_1<t_1<\cdots<s_k<t_k$. $f(s_i)\le a$. $f(t_i)\ge b$ (If every $k\in\mathbb{Z}^+$. it hold. Then $M^f_{ab}(I)=\infty$). where $s_i, t_i\in I$.

Lemma. D countable dense set of $\mathbb{R}^+$. $f: D \to \mathbb{R}$.

If i) f is bdd in $D \cap [0,T]$. $\forall T \in D$.

ii) $\forall a < b$. $a, b \in \mathbb{Q}$. $M_{ab}^{f}(D \cap [0,T]) < \infty$. $\forall T \in D$.

Then: $f(t+) = \lim_{s \downarrow t, s \in D} f(s)$. $f(t-) = \lim_{s \uparrow t, s \in D} f(s)$ exist for

$\forall t > 0$. So $g(t) = f(t+)$ is càdlàg

Thm. (X_t) is supermart. $D \subseteq \mathbb{R}^+$. countable dense.

i) For P-a.s. $w \in \Omega$. $s \mapsto X_s(w)$ defined on D

has left and right-limit. i.e. $X_{t+}(w) = \lim_{s \downarrow t, s \in D} X_s(w)$

$X_{t-}(w) = \lim_{s \uparrow t, s \in D} X_s(w)$. exists. $\forall t \in \mathbb{R}^+$.

ii) $\forall t \in \mathbb{R}^+$. $X_{t+} \in L^1$. $X_t \geq E(X_{t+} | \mathcal{F}_t)$.

If $t \mapsto E(X_t)$ is right-conti. Then: $X_t = E(X_{t+} | \mathcal{F}_t)$

iii) (X_{t+}) is supermart. w.r.t. $(\mathcal{F}_{t+})$. Moreover.

it's mart. if (X_t) is mart.

Pf: i) To use Lemma. First, $\sup_{0 \le s \le T} |X_s| < \infty$. a.s. by Rmk.

and $E((X_T - a)^-)/(b-a) \geq E(M_{ab}^X(D_n)) \uparrow E(M_{ab}^X(D^T))$

where D_n is finite subset of $D \uparrow D \cap [0,T] =: D^T$.

$\Rightarrow M_{ab}^X(D \cap [0,T]) < \infty$. a.s. $\forall T \in D$.

So: $N = \bigcup_{T \in D} \left(\{\sup_{t \in D \cap [0,T]} |X_t| = \infty\} \cup \left(\bigcup_{a, b \in \mathbb{Q}, a < b} \{M_{ab}^X(D \cap [0,T]) = \infty\} \right) \right)$

is a P-null set.

ii) Def: $X_{t+}(w) = \begin{cases} \lim_{s \downarrow t, s \in D} X_s(w) & \text{if the lim. exists.} \\ 0. & \text{otherwise} \end{cases}$

So it's also $\mathcal{F}_{t+}$-measurable

By construction: $X_{t+} = \lim_n X_{t_n}$. $(t_n) \subset D \downarrow t$.

Set $Y_k = X_{t-k}$. $k \le 0$. is backward supermart.

since $\sup_k E|Y_k| < \infty$. $\Rightarrow Y_k \to X_{t+} \in L^1$.

Besides. $X_t \ge E(X_{t_n} | \mathcal{F}_t) \xrightarrow{n \to \infty} E(X_{t+} | \mathcal{F}_t)$

If $E(X_t) = E(X_{t+})$. RHS $= E(E(X_{t+}|\mathcal{F}_t)) \le E(X_t)$

$\Rightarrow X_t = E(X_{t+} | \mathcal{F}_t)$.

iii) For $s < t$. $s_n \downarrow s$. $s_n \le t_n$. $\forall n$. $X_{s_n} \xrightarrow{L^1} X_{s+}$.

consider $\forall A \in \mathcal{F}_{s+}$. $\lim_n E(X_{s_n} I_A) \ge \lim_n E(t_n I_A)$

$= E(X_{t+} I_A) = E(E(X_{t+} | \mathcal{F}_{s+}) I_A)$

$\Rightarrow X_{s+} \ge E(X_{t+} | \mathcal{F}_{s+})$. "$=$" holds if X is mart.

<u>Thm</u>: If $(\mathcal{F}_t)$ is right-conti. complete. (X_t) is supermart.

st. $t \mapsto E(X_t)$ is right-conti. Then X has a modification with càdlàg sample path. is $\mathcal{F}_t$-supermart.

<u>Pf</u>: Set $Y_t(\omega) = \begin{cases} X_{t+}(\omega). & \text{if } \omega \notin N. \\ 0. & \text{if } \omega \in N \end{cases}$ N is in Thm above.

since $X_{t+} \in \mathcal{F}_{t+} = \mathcal{F}_t$. $\Rightarrow X_t = E(X_{t+} | \mathcal{F}_t) = X_{t+} = Y_t$. a.s.

Y_t is càdlàg (By Lemma) modification. $\mathcal{F}_t$-supermart.

(4) Optional Stopping Thm:

<u>Thm</u>. If X is right-conti. supermart. $\sup_t E|X_t| < \infty$.

Then $\exists X_\infty \in L^1$. st. $X_t \to X_\infty$ a.s. $(t \to \infty)$.

Pf: As we have proved: $E(M_{ab}^{X}(D\cap[0,T])) \le \sup_t E(X_t-a)^-/(b-a)$

where $D \subset \mathbb{R}^+$ countable, dense. Set $T\to\infty$. $\therefore M_{ab}^{X}(D)<\infty$ a.s.

$\therefore X_\infty = \lim_{\substack{t\to\infty \\ t\in D}} X_t \in \overline{\mathbb{R}}$ exists. By Fatou's: $E|X_\infty| \le \underline{\lim}_{t\in D} E|X_t|$

Remove "$t\in D$" by right-conti of X_t.

Def: Mart. (X_t) is closed if $\exists Z \in L^1$. st. $X_t = E(Z|\mathcal{F}_t)$. $\forall t\ge 0$.

Thm. (X_t) is right-conti. mart. Then: follows equi.:

i) X is closed ii) X is u.i.

iii) X_t converges a.s. and L^1 as $t\to\infty$. $\exists X_\infty \in L^1$.

st. $X_t \to X_\infty$ a.s. $X_t = E(X_\infty|\mathcal{F}_t)$.

Pf: i) $\Rightarrow$ ii) $\Rightarrow$ iii) Same in discrete case.

iii) $\Rightarrow$ i): $X_s = E(X_{t_n}|\mathcal{F}_s) \to E(X_\infty|\mathcal{F}_s)$. for $t_n \uparrow \infty$.

Eg. SBm $(B_t)_{t\ge0}$ isn't u.i.: $E(|B_t| I_{\{|B_t|>m\}}) = \int_m^\infty \frac{2x}{\sqrt{2\pi t}} e^{-x^2/2t}dx$

$= 4\sqrt{\frac{t}{2\pi}}\, e^{-m^2/2t} \to \infty$. as $t\to\infty$.

Thm. (X_t) is right-conti. u.i mart. $S\le T$ are two stopping time. Then $X_S, X_T \in L^1$. $X_S = E(X_T|\mathcal{F}_S)$

Pf: $T_n = \frac{[2^nT]+1}{2^n}$. $S_n = \frac{[2^nS]+1}{2^n}$. discreten.

Consider $Y_k^{(n)} = X_{k/2^n}$ discrete mart. w.r.t. $\mathcal{H}_k^{(n)} = \mathcal{F}_{k/2^n}$.

$\Rightarrow X_{S_n} = Y^{(n)}_{2^nS_n} = E(Y^{(n)}_{2^nT_n}|\mathcal{H}^{(n)}_{2^nS_n}) = E(X_{T_n}|\mathcal{F}_{S_n})$

For $A\in\mathcal{F}_S \subset \mathcal{F}_{S_n}$ $\Rightarrow$ $E(X_{S_n}I_A) = E(X_{T_n}I_A)$

Set $n\to\infty$. by right-conti and u.i: $E(X_SI_A) = E(X_TI_A)$

Besides $X_S = X_S I_{\{S<\infty\}} + X_\infty I_{\{S=\infty\}} \in \mathcal{F}_S$.

<u>Cor.</u> (X_t) is right-conti mart. $S \le T \le n$. two bdd stopping times. Then. $X_S, X_T \in L^1$. $E(X_T | \mathcal{F}_S) = X_S$.

<u>Pf:</u> Apply to $(X_{t\wedge n})$ closed by X_n.

<u>Cor.</u> (X_t) is right-conti mart. T is stopping time.

i) $(X_{t\wedge T})$ is still mart.

ii) (X_t) is u.i. $\Rightarrow$ $(X_{t\wedge T})$ is u.i.

and $\forall t \ge 0$. $X_{t\wedge T} = E(X_T | \mathcal{F}_t)$.

<u>Pf:</u> ii) $t\wedge T \le T$ $\Rightarrow$ $X_T, X_{T\wedge t} \in L^1$.

and $E(X_T | \mathcal{F}_{t\wedge T}) = X_{T\wedge t}$. by Thm.

For $A \in \mathcal{F}_t$. $A \cap \{T>t\} \in \mathcal{F}_t \cap \mathcal{F}_T = \mathcal{F}_{T\wedge t}$

$\Rightarrow E(X_T I_{\{T>t\}\cap A}) = E(X_{T\wedge t} I_{\{T>t\}\cap A})$

Besides. $E(X_T I_{\{T\le t\}\cap A}) = E(X_{T\wedge t} I_{\{T\le t\}\cap A})$

So: $E(X_T I_A) = E(X_{T\wedge t} I_A)$ $\forall A \in \mathcal{F}_t$.

i) For $n>0$. apply ii) on $(X_{t\wedge n})$. u.i.

<u>Thm.</u> $(Z_t)_{t\ge 0}$ nonnegative right-conti supermart. If $U \le V$. two stopping time. Then. $Z_U, Z_V \in L^1$ and $E(Z_V | \mathcal{F}_U) \le Z_U$.

<u>Rmk:</u> i) Note: $E(Z_t) \le E(Z_0)$. $\forall t$. $\Rightarrow \exists Z_\infty \in L^1$. s.t. $Z_t \to Z_\infty$. a.s and in L^1.

ii) It implies: $E(Z_V) \le E(Z_U)$. So:

$$E(Z_U I_{\{U<\infty\}}) \ge E(Z_V I_{\{U<\infty\}}) \ge E(Z_V I_{\{V<\infty\}})$$

Pf: 1°) First, assume $U \le V \le p$. $\exists\, p$ const.

Set $U_n = \frac{[2^n U]+1}{2^n}$, $V_n = \frac{[2^n V]+1}{2^n}$. (discretize)

By right-conti, $Z_{U_n} \to Z_U$ a.s. $Z_{V_n} \to Z_V$ a.s.

Apply discrete optional stopping Thm. on $Z_{k/2^{n+1}}$

$\Rightarrow E(Z_{U_n} | \mathcal{F}_{U_{n+1}}) \le Z_{U_{n+1}}$. $\forall n \ge 0$.

Set $Y_n = Z_{U_{-n}}$. backward supermart. w.r.t. $(\mathcal{F}_{U_n})$.

Since $E(Z_{U_n}) \le E(Z_0) \Rightarrow Z_{U_n} \xrightarrow{L^1} Z_U \in L^1$.

Similarly. $Z_{V_n} \xrightarrow{L^1} Z_V$. combined with $E(Z_{U_n}) \ge E(Z_{V_n})$.

2°) To remove "$U \le V \le p$".

First, note: $E(Z_{U\wedge p}) \le E(Z_0)$. $\xRightarrow{\text{Fatou's}}$ $E(Z_U) \le E(Z_0)$.

Fix $A \in \mathcal{F}_U \subset \mathcal{F}_V$. $U^A = \begin{cases} U & \text{if } w \in A \\ \infty & \text{if } w \notin A \end{cases}$ is stopping time.

By 1°): $E(Z_{U^A \wedge p}) \ge E(Z_{V^A \wedge p})$

$\Rightarrow E(Z_U I_{A \cap \{U \le p\}}) \ge E(Z_V I_{A \cap \{U \le p\}})$

By Monotone Converge Thm: let $p \to \infty$.

Combined with $E(Z_U I_{A \cap \{U=\infty\}}) = E(Z_V I_{A\cap\{V=\infty\}})$

So: $E(Z_U I_A) \ge E(Z_V I_A) = E(E(Z_V | \mathcal{F}_U) I_A)$