① Strong Stochastic DCT:

Def: $M \in \mathcal{M}^c_{loc}$. $H^{(n)}, H \in L^2_{loc}(M)$. We say $H^{(n)} \to H$ in $L^2_{loc}(M)$ if $\forall \varepsilon > 0$, $\forall T > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\int_0^T |H^{(n)}_t - H_t|^2 d\langle M\rangle_t > \varepsilon \right) = 0.$$

Lemma For $M \in \mathcal{M}^c_{loc}$. Then: $H^{(n)} \to H$ in $L^2_{loc}(M)$

$\iff \int_0^\cdot H^{(n)} dM \to \int_0^\cdot H dM$ ucp.

Thm. (Stochastic DCT)

X is conti. semimart. $H^{(n)}, H$ are progressive and $H^{(n)}_t \xrightarrow{a.s.} H_t$, $\forall t \geq 0$. And $\sup_{[0,T]} |H^{(n)}_t(\omega)| \leq C_T(\omega) < \infty$, $\forall T > 0$. Then:

$H^{(n)}, H \in \mathbb{L}(X)$. $\int_0^\cdot H^{(n)}_s dX_s \xrightarrow{ucp} \int_0^\cdot H_s dX_s$.

Pf: Set $k_t(\omega) = \sum C_n(\omega) I_{[n-1,n)}(t)$

$k_t \geq \sup_{[0,t]} |H^{(n)}_s|$, $\forall t$.

$$\int_0^\cdot H^{(n)}_s dX_s = \int_0^\cdot H^{(n)}_s dM_s + \int_0^\cdot H^{(n)}_s dA_s$$

By usual DCT. for $\forall \omega \in \Omega$.

$$\int_0^\cdot H_s^{(n)} dA_s \to \int_0^\cdot H_s dA_s. \quad \mathbb{P}\text{-a.s.}$$

For the former one:

$$\left\langle \int_0^\cdot H_s^{(n)} - H_s \, dM_s \right\rangle_t = \int_0^t |H_s^{(n)} - H_s|^2 d\langle M\rangle_s$$

So: Apply usual DCT for $w \in \Omega$.

$\Rightarrow \langle \;\; \rangle \xrightarrow{a.s} 0$. So: $\int_0^t H_s^{(n)} dM \xrightarrow{ucp} (H\cdot M)_t$

Rmk: $K_t(w)$ defined here isn't progressive

But we don't need it. (*)

Cor. X is conti. semimart. $H^{(n)}$, H are càdlàg adapted st. $H^{(n)} \xrightarrow{ucp} H$.

Then: $\int_0^\cdot H^{(n)} dX_s \xrightarrow{ucp} \int_0^\cdot H \, dX_s$

Pf: $C_T(w) = \sup_{[0,T)} \left(\sup_n |H_t^{(n_k)}| + |H_t| \right)$

$< \infty$. Since continuity.

where $H_t^{(n_k)} \xrightarrow{acc} H_t$. $\mathbb{P}$-a.s.

with subseq. convergence argm(*).

(*): (X_n) conv. $\Leftrightarrow$ $\forall (X_{n_k}) \subset (X_n)$. $\exists (X_{n_k'}) \subset (X_{n_k})$ conv. in (X, d)

For (*): Sometimes we use some trick to let r.v. to be progressive:

prop. τ is stopping time. $h \in \mathcal{G}_\tau$. Set

$H := h I_{[\tau,\infty)}$ $\Rightarrow$ $\forall X$ is conti. semi-mart. $H \in \mathbb{L}(X)$ and $\int_0^t H_s dX_s = h \cdot (X_t - X_{t\wedge\tau})$

Rmk: $h I_{[0,\tau)}$ isn't progressive

But $h I_{[\tau,\infty)}$ can!

Pf: By localization: Set h is bdd and $X \in \mathcal{M}_c^2$

Set $\tau_n := \frac{[2^n \tau]}{2^n} \in [t_k^n = \frac{k}{2^n}]$.

$\Rightarrow h^n := h I_{[\tau^n,\infty)}$

$= \sum_k h I_{\{\tau^n = t_k^n\}} I_{[t_k^n,\infty)}$

$\triangleq \sum \tilde{h} (1 - I_{[0,t_k^n)})$

$h \in \mathcal{G}_{t_k^n}$. And inside $\in b\mathcal{E}$.

So: $\int h^n dM_s = \sum \tilde{h} (M_t - M_{t\wedge t_k^n})$

By Stochastic DCT. $|h^n| \le |h|$

$\Rightarrow \int h^n dM_s \to \int h dM_s$

$= h (M_t - M_{\tau\wedge t})$.

And let $\tau_n \to \infty$. localization.

① Prop. $M \in \mathcal{M}^c_{loc}$. Then. a.s. $w \in \Omega$. We have

$M_r = M_s$. $\forall r \in [s,t] \Leftrightarrow \langle M\rangle_s = \langle M\rangle_t$

Pf: Note $\langle M\rangle$. M. are both consi.

We only inspect $q \in [0,t) \cap \mathbb{Q}$.

$\tau_q := \inf\{s \geq q \mid M_s - M_q \neq 0\}$

$\sigma_q := \inf\{s \geq q \mid \langle M\rangle_s - \langle M\rangle_q \neq 0\}$.

prove: $\tau_q = \sigma_q$. $\forall q \in \mathbb{Q}$

It from $\forall q \geq p$. we know:

$\langle M^q - M^p\rangle = \langle M\rangle^q - \langle M\rangle^p$.

Let $q = \tau_p$ or σ_p. We obtain

$\sigma_p \geq \tau_p$ (from $\Rightarrow$). $\tau_p \geq \sigma_p$ (from $\Leftarrow$)

② For (H_s) càdlàg adapted. We have:

$\mathbb{P}(\int_0^\cdot H_s dB_s \in C^\alpha([0,T]) . \forall \alpha < 1/2 . \forall T > 0) = 1$

Pf: ~~Set~~ $\tau_n = \sup\{t \geq 0 \mid |H_t| \geq n\}$

$\tau_n \uparrow \infty$ as $n \to \infty$ since H_s is

locally bdd. LHS $\leq \mathbb{P}(\square, \tau_n \geq t) +$

$\underset{\to 0}{\mathbb{P}(\tau_n < t)}$. Next, consider on $\{\tau_n \geq t\}$.

Set $M_t = \int_0^t H_s I_{[u,v]} \, dB_s \in \mathcal{M}^{loc}$.

Apply BDG & Hölder inequi. with Kolmogorov's criteria. We got:

$E(\| \int_0^\cdot H_s \, dB_s \|^p_{C^\alpha([0,T])}) \le C p^p$.

Rmk: Using $\int H dB$, we can construct a counterexample for statement "$\mathcal{M}^c_{loc} \subset C^\alpha(\mathbb{R}^1)$, $\forall \alpha < \frac{1}{2}$."

Set $H_s = (\beta t^{\beta-1})^{\frac{1}{2}}$. We have

$\langle \int_0^\cdot H_s \, dB_s \rangle_t = t^\beta \to \infty$. $\frac{1}{2} < \beta < 1$

$\Rightarrow \int_0^t H_s \, dB_s = B_{t^\beta} \in C^{\beta(\frac{1}{2}-\epsilon)}([0,1])$

④ Thm. If X is local mart. $\{\tau_t\}_{t\ge 0}$ is seq of finite stopping times. s.t. $t \mapsto \tau_t$ is increasing. conti. $\Rightarrow \tilde{X}_t := X_{\tau_t}$ is local mart w.r.t $\mathcal{G}_t := \mathcal{F}_{\tau_t}$.

Rmk: Set $\tau_t = f(t)$. Cantor func.

So $\{\tau_t\}$ is increasing, conti. but not AC. $\tilde{B}_t = B_{\tau_t} \in \mathcal{M}^{loc}$

($B_{f(t)}$ is mart. to $\mathcal{F}_{f(t)}$ in fact.)

But $\langle \widetilde{B} \rangle_t = 2t \notin AC[0,1]$.

So, although QV of $\mathcal{M}_c^{loc}$ is

conti. ↑. 1-var. It's not AC

necessarily! (Recall. ∀ increasing

func = AC + conti. + jump.)

(5) Conclusion under complex-value:

Def: $X = X^1 + iX^2$ is $\mathbb{C}$-valued semimart. if

(X^1, X^2) is $\mathbb{R}^2$-value semimart.

Rmk: $\langle\ ,\ \rangle$ is still sym. bilinear. And

$\langle zM, N \rangle = z\langle M, N \rangle$. $\forall z \in \mathbb{C}$.

i) For f holomorphic. $\mathbb{C}$-value Itô formula

still holds: $df(X_t) = f'(X_s)dX_s + \frac{1}{2}f''(X_s)d\langle X \rangle_s$

ii) For $\mathbb{C}$-value $\mathbb{F}$-adapted conti. X_t. s.t.

$X_0 = 0$. X is $\mathbb{C}$-valued $\mathbb{F}$-BM $\iff$

$X \in \mathbb{C}\text{-}\mathcal{M}_c^{loc}$ & $\langle X \rangle_t = 0$, $\langle X, \overline{X} \rangle_t = 2t$.

iii) M is $\mathbb{C}$-$\mathcal{M}_{loc}^c$ with $M_0 = 0$. If $\langle M \rangle_t = 0$

& $\langle M, \bar{M} \rangle_\infty = \infty$ a.s. Then $\exists$ $\mathbb{C}$-BM β. s.t. $M_t = \beta_{\langle M, \bar{M} \rangle_t / 2}$ a.s.

<u>Cor.</u> (Conformal invar. of $\mathbb{C}$-BM)

f is holomorphic $\Rightarrow$ $\langle f(B) \rangle_t = 0$.

$\langle f(B), \overline{f(B)} \rangle_t = 2 \int_0^t |f'(B_s)|^2 ds$

(So. $\exists$ β. $\mathbb{C}$-BM. $f(B_t) = f(B_0) + \beta_{\int_0^t |f'(B_s)|^2 ds}$. $\forall t < \int_0^\infty |f'(B)|^2$.)

⑥ Recurrence of BM: B is d-dim BM

i) $d = 1$. B is point-recurrent, but not positive recurrent.

ii) $d = 2$. $\mathbb{P}(\exists t. B_t = x) = 0$. $\forall x \in \mathbb{R}^2 / \{B_0\}$

But it's nbd recurrent. i.e.

$\mathbb{P}(\forall r > 0. \forall x \in \mathbb{R}^2, B_t \in B(x, r), \text{i.o}) = 1$

iii) $d \geq 3$. B is transient: $\lim_{t \to \infty} |B_t| = \infty$

<u>Pf:</u> i) Prove $\int_0^\infty e^{2B_t^i} dt = +\infty$. $i = 1, 2$.

by Kolmogorov 0-1 law. (i.e.

$\Leftrightarrow \bigcap_n \{ \int_n^\infty e^{2B_t^i} dt \geq 1 \} \in \mathcal{F}$ has

full measure.

Note $\mathbb{P}(\bigcap_n \square) \geq \mathbb{P}(\bigcap_n \{ e^{2B_n^i} \int_0^1 e^{2\beta_t} dt > 1 \})$

$\geq \mathbb{P}(\bigcap \{ B_n > 0. \int_0^1 e^{2\beta_t} > 1 \})$

$= \mathbb{P}(B_1 > 0, \int_0^1 e^{2\beta_t} > 1) > 0.)$

So for $f(z) = z_0 (1 - e^z)$. $\forall z_0 \in \mathbb{C}$.

We have: $\int_0^\infty | f'(B_t) |^2 dt = \infty$.

By DDS repre: $f(B_t) = \tilde{\beta}_{\square_t}$.

$\mathbb{P}(\exists t > 0. \tilde{\beta}_t = z_0) = \mathbb{P}(\exists t. \tilde{\beta}_{\square_t} = z_0)$

$= \mathbb{P}(\exists t > 0. f(B_t) = z_0) = 0.$

follow from $f(z) = z_0$ has no solution

iii) Check. $\forall 1 < p < d$. for $Z \sim N(0, I_d)$

$\sup_{x_0} \mathbb{E}(1 / |x_0 + z|^p) < \infty$ (split $|x_0 + z| \leq R$)

$\sup_{\lambda} \mathbb{E}(1 / |\lambda z + x_0|^p) < \infty$ (split $|x_0 + \lambda z| \leq \lambda R$)

And note that $|x_0 - B_t|^{2-d} =: M_t \in \mathcal{M}_c^{loc}$.

So M_t is also supermart.

With observation above. $\forall\, q < d/d-2$.

We have $\sup_t \mathbb{E}(|M_t|^q) < \infty$. and

$\mathbb{E}(|M_t|^q) \to 0$ $(t \to \infty)$. So $M_t \to 0$ a.s.

i.e. $|D_t| \xrightarrow{a.s.} +\infty$ $(t \to \infty)$.

(7) On Cameron - Martin formula:

Note here we require $h \in L^2(\mathbb{R}^+; \mathbb{R}^d)$

deterministic func. When $h = h(w,t)$ also

involves randomness, if let $\mathbb{E}(e^{\frac{1}{2}\langle h \cdot D\rangle_\infty}) < \infty$

then $\mathcal{E}(h \cdot B)$ is still u.i. mart. (Novikov's)

And we can set $dQ/dP = \mathcal{E}(-h \cdot B)$. So:

$B + \int h\,ds \in Q\text{-}\mathcal{M}_c^{loc} \xRightarrow{Levy} B + \int h\,ds \sim BM$.

Rmk: i) It can be seen: $B + F$. shift by $F \in W^{1,2}$.

ii) For $h \in L^2_{loc}(\mathbb{R}^+; \mathbb{R}^d)$. e.g. $h \equiv \theta$.

CM formula doesn't work!

But $B_t + \theta t$ & B_t has same dist.

under diff. p.m. on $[0,T]$. $\forall T > 0$.

by set $h = \theta I_{[0,T]} \in L^2$. But at ∞:

$B_t + \theta t \xrightarrow{t\to\infty} +\infty$. while $\varliminf B_t = -\infty$

⑧ Backward & forward eq.:

$\mathcal{L}$ is diffusion generator. $\varphi: \mathbb{R}^d \to \mathbb{R}^1$. locally bdd & measurable. Kolmogorov backward eq. is

$\partial_t u(t,x) = \mathcal{L} u(t,x)$, $u(0,x) = \varphi(x)$.

Thm. i) $\varphi \in C_B^2 \Rightarrow u(t,x) = \mathbb{E}(\varphi(X_t^x))$ unique

solve backward eq. where X_t^x is weak

s.l. of SDE with datum x

ii) $\exists u \in C_B^{1,2}$ solves backward eq. & $\exists$ weak

s.l. X_t^x for SDE $\Rightarrow u(t,x) = \mathbb{E}(\varphi(X_t^x))$

Cor. $\sup_{t,x} u(t,x) \leq \sup_x \varphi(x)$. maximal prin.

Rmk: In Feynman-Kac repre. we have

same argument: for $c \in C_B$. $u \in C_B^{1,2}$

solves $\partial_t u = \mathcal{L}u + cu$, $u(0,x) = \varphi(x)$. $\Rightarrow$

u is uniquely $= \mathbb{E}(\varphi(X_t^x) e^{\int_0^t c(X_t^x) dt})$

conversely, require $\varphi \in C_B^2$. $u(t,x)$

$= \mathbb{E}(\varphi(X_t^x) e^{\int \cdots})$ will solve the PDE.

Def: Kolmogorov forward equation is the FPE:

$\partial_t \mu = \mathcal{L}^* \mu$. $\mu_0 = \zeta$. $\mathcal{L}^*$ is dual of $\mathcal{L}$.