Conti. Semimartingales.

(1) Finite Variation Process:

Next, we consider real-valued processes indexed by $\mathbb{R}^+$.

① Function:

Def: $T \geq 0$. $a: [0,T] \to \mathbb{R}^1$. conti. $a(0) = 0$. is finite variation if $\exists$ sign measure μ on $[0,T]$.

s.t. $a(t) = \mu[0,t]$. $\forall t \in [0,T]$.

Rmk: μ is uniquely determined. Besides. $\mu \ll \mathcal{L}$.

prop. $a \in FV[0,T] \iff a$ is two monotone nondecreasing conti. functions vanishing at 0.

Pf: ($\Rightarrow$) By Jordan Decomposition ($\Leftarrow$) Trivial.

For $f: [0,T] \to \mathbb{R}^1$. measurable. s.t. $\int_{[0,T]} |f| \, |\mu|(ds) < \infty$.

Set $\int_0^T f(s)\, da(s) = \int_{[0,T]} f(s)\, \mu(ds)$. $\int_0^T f(s)\, |da(s)| = \int_{[0,T]} f\, |\mu|(ds)$.

$\Rightarrow$ We can define $\int_0^t f(s)\, da(s)$ for $\forall\, 0 \leq t \leq T$ by restrict.

Besides $t \mapsto \int_0^t f(s)\, da(s) \in FV[0,T]$ with $\tilde{\mu} = f\, \mu(ds)$

prop: (Total Variation)

$\forall t \in [0,T]$. $\int_0^t |d\mu(s)| = \sup\{\sum_1^p |\mu(t_i) - \mu(t_{i-1})| \mid 0 = t_0 < t_1 < \cdots < t_p = t$ is subdivision of $[0,t]\}$.

Moreover. $\forall$ increasing seq : $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ subdivision whose mesh $\to 0$. We have : $\int_0^t |d\mu(s)| = \lim_{n\to\infty} \sum_1^{p_n} |\mu(t_i^n) - \mu(t_{i-1}^n)|$.

<u>Pf</u>: WLOG. Set $t = T$. (by restriction).

1') Note: $|\mu(t_i) - \mu(t_{i-1})| = |\int_{t_{i-1}}^{t_i} d\mu(s)| \le |\mu|(t_i, t_{i+1}]$

2') For conversed inequi. Prove the second assertion

Consider $([0,T], \mathcal{B}_{[0,T]}, P)$. $P(ds) = \frac{|\mu|(ds)}{|\mu|[0,T]}$

Filtration $\mathcal{B}_n = \sigma((t_{i-1}^n, t_i^n], 1 \le i \le p_n)$.

$X(s) = \frac{d\mu}{d|\mu|}(s) = I_P - I_N$. Set $X_n = E(X \mid \mathcal{B}_n)$.

$\Rightarrow X_n = \sum_1^{p_n} \frac{\mu(t_{i-1}^n, t_i^n]}{|\mu|(t_{i-1}^n, t_i^n]} I_{(t_{i-1}^n, t_i^n]}$ closed mart. w.r.t. $\mathcal{B}_n$.

So $X_n \xrightarrow{L^1} X$. $\Rightarrow \lim_n E|X_n| = E|X| = 1$. $(X \in \mathcal{B} = \vee \mathcal{B}_n)$

Combined with : $E|X_n| = \sum_1^{p_n} |\mu(t_i^n) - \mu(t_{i-1}^n)| / |\mu|[0,T]$.

<u>Lemma.</u> If $f: [0,T] \to \mathbb{R}^1$ conti. $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = T$ subdivision of $[0,T]$. St. mesh $\to 0$. Then. we have:

$\int_0^T f(s)\, d\mu(s) = \lim_{n\to\infty} \sum_1^{p_n} f(t_{i-1}^n)(\mu(t_i^n) - \mu(t_{i-1}^n))$

<u>Pf</u>: By DCT. directly. $f_n(s) = \sum_1^{p_n} f(t_{i-1}^n) I_{(t_{i-1}^n, t_i^n]}$.

<u>Rmk</u>: We say $\mu \in FV[0,\infty)$ if $\forall T \ge 0$. $\mu \in FV[0,T]$.

For real Borel Function f on $\mathbb{R}_+$. st. $\int_0^\infty |f||d\mu(s)| < \infty$. Then. define : $\int_0^\infty f(s)\, d\mu(s) = \lim_{T\to\infty} \int_0^T f(s)\, d\mu(s)$

③ Process:

Def: In $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. adapted process $(A_t)_{t\geq 0}$ is called finite variation process if all its sample paths are FV on $\mathbb{R}^+$.

In addition, if the sample path are nondecrease. Then we call it increasing process.

Rmk: We can define finite variation process with càdlàg sample paths.

Lemma. Finite variation process $\Leftrightarrow$ difference of two increasing process

Pf: $(\Rightarrow)$ $V_t = \int_0^t |dA_s|$ is an increasing process.

$$A_t = \frac{1}{2}(V_t + A_t) - \frac{1}{2}(V_t - A_t).$$

prop. For (A_t) finite variation process. H is progressive process. s.t. $\forall t\geq 0$. $\forall \omega \in \Omega$. $\int_0^t |H_s(\omega)||dA_s(\omega)| < \infty$.

Then $(H\cdot A)_{t\geq 0}$ the process defined by:

$(H\cdot A)_t = \int_0^t H_s \, dA_s$ is also finite variation process.

Pf: 1°) $\int_0^t H_s(\omega) \, dA_s(\omega) \in FV(\mathbb{R}^+)$.

2°) Check $(H\cdot A)_{t\geq 0}$ is adapted.

It's verified by the following Lemma.

Lemma. For $t>0$ fixed. If $h: \Omega \times [0,t] \to \mathbb{R}^1$ is $\mathcal{F}_t \otimes \mathcal{B}_{[0,t]}$ - measurable. $\int_0^t |h(w,s)||dA_s(w)| < \infty$ for $\forall w \in \Omega$. Then: $\int_0^t h(w,s)\, dA_s(w) \in \mathcal{F}_t$.

Pf: 1°) For $h(w,s) = I_{(u,v]}(s) I_{\Gamma}(w)$. $\Gamma \in \mathcal{F}_t$.

2°) By MCT, It holds for $h = I_G$. where $G \in \mathcal{F}_t \otimes \mathcal{B}_{[0,t]}$ as well. from 1°)

3°) For general h which is pointwise limit of seq of simple func. by DCT.

Rmk: i) For assumption: $P(\forall t \geq 0.\ \int_0^t |H_s(w)||dA_s(w)| < \infty) = 1$. If the filtration $(\mathcal{F}_t)$ is complete, then we can define $H \cdot A$ by $H' \cdot A$. where $H' = \begin{cases} H_t(w), & \text{if } \int_0^n |H_s||dA_s| < \infty.\ \forall n. \\ 0. & \text{otherwise.} \end{cases}$

is progressive.

ii) If: H, K progressive process. $\int_0^t |H_s||dA_s| < \infty$. $\int_0^t |H_s K_s||dA_s| < \infty$. for $\forall t \geq 0$. Then:

$K \cdot (H \cdot A) = (KH) \cdot A$. it's associable.

iii) In particular. consider $A_t = t$. We have $\int_0^t H_s\, ds$ is finite variation process.

(2) Conti Local Mart:

Consider in $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. filtrated Prob space.

Denote: $X_t^T = X_{T \wedge t}$. $\forall t \geq 0$. $X^T = (X_{t \wedge T})_{t \geq 0}$. for X.

Def: Adapted process (M_t) with conti sample paths is conti. local mart. If $\exists (T_n)$ nondecrease stopping times. s.t. $T_n(\omega) \uparrow \infty$. $\forall \omega$. for $\tilde{M} = M - M_0$ $\tilde{M}^{T_n}$ is u.i. mart. for $\forall n \in \mathbb{Z}^+$.

We say: (T_n) reduces M in this case.

Rmk: i) $M_t \in L^1$ is not required in def.

ii) It can define on cadlag paths.

iii) Conti. mart. is conti local mart.

Since $T_n = n$ reduces it. But the converse is false. e.g. $1/|\vec{B_t}|$. in $\mathbb{R}^N$. $N \geq 3$.

prop. For $Z \in \mathcal{F}_0$. M_t is c.l.m. Then: $N_t = Z M_t$ is conti. local. mart.

Pf: Let $T_n = \inf\{t \mid |N_t| \geq n\}$.

$\exists Z_k \in \mathcal{F}_0$. simple func. $|Z_k| \leq |Z|$. $Z_k \to Z$.

Suppose $(\tilde{T}_n)$ reduces M. Since $\tilde{T}_n \uparrow \infty$.

$\exists m_n$. st. $P(T_n \leq \tilde{T}_{m_n}) = 1$.

(Only consider $T_n < \infty$ a.s. Otherwise. $\exists n$. $T_n = \infty$ a.s. then N_t is bdd. a.s. trivial)

$\Rightarrow M^{T_n \wedge \tilde{T}_{m_n}} = M^{T_n}$ is u.i. $N_t^{T_n} \in L^1$. $\forall t$.

$\forall s < t$. $A_s \in \mathcal{F}_s$. $E(Z M_t I_{A_s}) = \lim_k E(\sum \lambda_i^k I_{A_i^k} M_t I_{A_s})$

$= \lim_k E(M_s^{T_n} \sum \lambda_i^k I_{A_i^k \cap A_s}) = E(M_s^{T_n} Z I_{A_s})$

Lemma. (M_t) is c.l.m. If (T_n) reduce M. (S_n) is seq of stopping times. $S_n \uparrow \infty$. Then $T_n \wedge S_n$ reduces M.

Pf: $M^{T_n \wedge S_n} = (M^{T_n})^{S_n}$. u.i.

Cor. The span of c.l.m.'s is a linear space.

Pf: T_n reduce M_t. T_n' reduce M_t'

$\Rightarrow T_n \wedge T_n'$ reduces $M_t + M_t'$.

Rmk: For T stopping time. M is c.l.m. Then: M^T is c.l.m. In particular. Let $T = n$. So: $T_n \wedge n$ reduces M as well. Actually. the u.i. mart in the definition can be replaced by mart. since M^{T_n} is mart. $\Rightarrow M^{T_n \wedge n}$ is u.i. mart.

Prop. i) Nonnegative c.l.m M. st. $M_0 \in L^1$ is supermart.

ii) For c.l.m M. If $\exists Z \in L^1$. st. $|M_t| \le Z$. $\forall t \geq 0$. Then: M is u.i. mart.

iii) If c.l.m M_t is with $M_0 \in L^1$. Then $T_n = \inf\{t \mid |\tilde{M}_t| \geq n\}$ reduces M. $\tilde{M}_t = M_t - M_0$.

Pf: i) From $M_{s \wedge T_n} = E(M_{t \wedge T_n} \mid \mathcal{F}_s)$. Let $n \to \infty$. It follows from Fatou's Lemma. Besides. Note: $E(M_t) \le E(M_0) < \infty$.

$\Rightarrow M_t \in L^1$. $\forall t \geq 0$.

ii) By DCT. It's mart.

iii) M^{T_n} is $n + |M_0|$. by ii).

Rmk: i) For c.l.m M_t: M is u.i $\not\Rightarrow$ M is mart.

e.g. $M_t = 1/|B_t|$. is bdd in L^2. c.l.m.

where B_t is N-dim Brownian motion.

$N \geq 3$. But it's not mart.

ii) In particular. if $\sup_t \|X_t\|_\infty < \infty$. then

by ii) in prop. X_t is truly u.i. mart.

Thm. M is FV c.l.m $\Rightarrow$ $M_t = 0$. $\forall t \geq 0$. a.s.

Pf: $\tau_n = \inf\{t \geq 0 \mid \int_0^t |dM_s| \geq n\}$. stopping time $\uparrow \infty$

Set: $N = M^{\tau_n}$ $\Rightarrow$ $|N_t| = |M_{t\wedge\tau_n}| \leq \int_0^{t\wedge\tau_n} |dM_s| \leq n$.

So: N_t is bdd mart. Prove: $E(N_t^2) = 0$. $\forall t \geq 0$.

Note: $E(N_t^2) = \sum_1^p E(N_{t_i^k} - N_{t_{i-1}^k})^2$

$\leq \sum_1^p E(\sup_i |N_{t_i} - N_{t_{i-1}}| \sum |N_{t_i}^k - N_{t_{i-1}}^k|)$

for subdivision $0 = t_0^k < t_1^k < \cdots < t_p^k = t$. whose

mesh $\to 0$. $\Rightarrow$ $\lim_k E(\sup |N_{t_i}^k - N_{t_{i-1}}^k|) = 0$.

follows from DCT. conti of N.

Combined with: $\sum^p |N_{t_i}^k - N_{t_{i-1}}^k| \leq \int_0^{t\wedge\tau_n} |dM_s| \leq n$.

Finally. let $n \to \infty$. $\tau_n \to \infty$ $\Rightarrow$ $M_t = 0$. a.s.

(2) Quadratic Variation:

· $(\mathcal{F}_t)$ is complete in this section.

Thm. M_t is c.l.m. There exists an increasing process denoted by $(\langle M,M\rangle_t)$ which is uniquely up to indistinguishability. s.t. $M_t^2 - \langle M,M\rangle_t$ is c.l.m.

Besides. for fix $t>0$. $0=t_0^n < t_1^n \cdots < t_{p_n}^n = t$ is subdivision from refine. whose mesh $\to 0$. $\langle M,M\rangle_t = \lim_n \sum_1^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2$ in prob. We say $\langle M,M\rangle_t$ is quadratic variation of M.

Rmk: i) Quadratic variation is a kind of way to define "Variance" in stochastic process.

Or it can be viewed as extension of variation variation of functions.

In nutshell. it measures "randomness" of process.

ii) Important example: $\langle B,B\rangle_t = t$. for BM.

iii) $\langle M,M\rangle$ doesn't depend on M_0 initial value.

But only rely on the increment.

iv) Actually. $\exists$ seq of subdivision $(\tilde{t}_i^n)$. s.t.

$\langle M,M\rangle_t = \lim_n \sum_1^{p_n} (M_{\tilde{t}_i^n} - M_{\tilde{t}_{i-1}^n})^2$. a.s. holds.

Pf: 1°) Uniqueness: If A, A' are two increasing process satisfies the condition. Then we have:

$A - A' = (M_t^2 - A') - (M_t^2 - A) = 0$ since it's FV c.l.m.

2°) For existence. first consider $M_0 = 0$. M is bdd.

Suppose $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = K$ (fixed) is subdivision of $[0,K]$. s.t. its mesh $\to 0$ $(t_i^n) \subset (t_k^m)$ if $n \le m$.

Check $X_t^n = \sum_1^{p_n} M_{t_{i-1}^n} (M_{t_i^n \wedge t} - M_{t_{i-1}^n \wedge t})$ is bdd mart.

Note: $\sum_1^j (M_{t_i^n} - M_{t_{i-1}^n})^2 = M_{t_j}^2 - 2X_{t_j}^n$. $\forall j \in \{0, 1, \cdots p_n\}$.

3°) We want to prove: X_t^n converges.

<u>Lemma</u>. $E(X_k^m - X_k^n)^2 \to 0$ as $m, n \to \infty$.

<u>Pf</u>: 1°) Consider $E(X_k^n X_k^m) =$ (fix $n \le m$)

$\sum_{i,j} E(M_{t_{i-1}} M_{t_{j-1}} (M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}}))$

The only nonzero terms correspond i, j

must satisfies $(t_{j-1}^m, t_j^m] \subset (t_{i-1}^n, t_i^n]$

Write $i_{n,m}(j)$ equals index i. s.t. $(t_{j-1}^m, t_j^m]$

$\subset (t_{i-1}^n, t_i^n]$ for every $j \in \{0 \cdots p_m\}$.

Note: $M_{t_i^n} - M_{t_{i-1}^n} = \sum_{k: i_{n,m}(k) = i} (M_{t_k^m} - M_{t_{k-1}^m})$

$\Rightarrow E(X_k^n X_k^m) = \sum_{\substack{1 \le j \le p_m \\ i = i_{n,m}(j)}} E(M_{t_{i-1}^n} M_{t_{j-1}}^m (M_{t_j^m} - M_{j-1}^m)^2)$

follows from mart. property.

2°) By 1°) and mart. property:

$E(X_k^n - X_k^m)^2 = E(\sum_{j, i = i_{n,m}(j)} (M_{t_{i-1}} - M_{t_{j-1}^m})^2 (M_{t_j^m} - M_{t_{j-1}^m})^2)$

$\le E(\sup (M_{t_{j-1}} - M_{t_{i-1}^n})^4)^{\frac{1}{2}} E((\sum (M_{t_j^m} - M_{t_{j-1}^m})^2)^2)^{\frac{1}{2}}$

By conti of M and DCT. $E(\sup \square) \to 0$.

The second term is bdd. check by expand.

$\Rightarrow$ By Doob's Inequi. $\lim_{n,m} E(\sup_{t \le k} (X_t^n - X_t^m)^2) = 0$.

So: (X_t^n) is Cauchy in L^2 $\exists (n_k) \subset (n)$. s.t.

$E(\sup_{t \le k} (X_t^{n_{k+1}} - X_t^{n_k})^2) \le 2^{-k} \Rightarrow \sum \sup_{t \le k} |X_t^{n_{k+1}} - X_t^{n_k}| < \infty$. a.s.

Set $Y_t = \lim_k X_t^{n_k}$ on Ω/N. $Y_t = 0$ on $N = \{\sum \square = \infty\}$ p-null.

Y_t is also L^2-limit of X. $\Rightarrow Y^k$ is conti. mart.

4°) Note : $M_t^2 - 2X_t^n$ is nondecreasing on $t \in [0,k]$.

Along (n_k), it converges to $M_t^2 - 2Y_t \uparrow$ on $[0,k]$ a.s.

Set $A_t^{(k)} = \begin{cases} M_t^2 - 2Y_t, & \Omega \setminus N \\ 0, & N \end{cases}$ $\forall t \in [0,k]$.

$\Rightarrow A_t^{(k)} \in \mathcal{F}_t$, $\uparrow$, conti. $M_{t\wedge k}^2 - A_{t\wedge k}^{(k)}$ is mart.

5°) Consider $k = l$ along N. We have : $(A_t^{(l)})_{0 \le t \le l}$.

Define increasing process $\langle M, M\rangle_t = A_t^{(l)}$. $\forall\, 0 \le t \le l$, a.s.

It's well-def. since $A_{t\wedge l}^{(l+1)} = A_{t\wedge l}^{(l)}$ by uniqueness, a.s.

$\langle M, M\rangle_t = 0$. on $(\bigcup_l N_l) \cup (\bigcup_{k<l} [A_{t\wedge k}^{(l)} \neq A_{t\wedge k}^{(k)}])$ null set

6°) $A_{t\wedge k}^{(k)}$ is distinguishable from $\langle M, M\rangle_{t\wedge k}$. on $[0,k]$.

Note $X_k^n \xrightarrow{L^2} Y_k = \frac{1}{2}(M_k^2 - A_k^{(k)}) \Rightarrow \Sigma(\cdots)^2 \xrightarrow{L^2} A_k^{(k)}$.

So $\lim_n \sum_1^{p_n} (M_{t_i} - M_{t_{i-1}})^2 = \langle M, M\rangle_k$ in L^2.

7°) To remove the assumptions :

$M_t = M_0 + N_t$. $M_t^2 = M_0^2 + 2M_0 N_t + N_t^2$. $M_0 N_t$ is c.l.m.

So we can remove "$M_0 = 0$". by consider N_t.

Set $T_n = \inf\{t \ge 0 \mid |M_t| \ge n\}$. Apply argue on M^{T_n}.

$A^{[n]} = \langle M^{T_n}, M^{T_n}\rangle$. Indistinguishable with $(A^{[n+1]})^{T_n}$.

Construct $A_{t\wedge T_n} = A_t^{[n]}$ a.s $\forall n \in \mathbb{Z}$. as before.

Since $M_{t\wedge T_n}^2 - A_{t\wedge T_n}$ is mart. $\Rightarrow M_t^2 - A_t$ is c.l.m.

8°) Note $T_n \uparrow \infty$. $\Rightarrow P(t \le T_n) \to 1$ $(n \to \infty)$.

Then we can consider $M_0 = 0$. and M^{T_m} :

$P(|\langle M, M\rangle_t - \sum_1^{p_n} (\cdots)^2| \ge \varepsilon) \le P(t \le T_m, |\langle M, M\rangle_t - S_n| \ge \varepsilon)$

$+ P(t > T_m) \to 0$ as $n, m \to \infty$.

That's why "converge in prob." comes in.

Prop. M is c.l.m. T is stopping time. Then we have:

$<M^T, M^T>_t = <M, M>_{t\wedge T}$ a.s. $\forall t$.

Pf: By uniqueness.

Prop. M is c.l.m. Then: $<M, M> = 0 \iff M_t = M_0$ a.s. $\forall t \geq 0$.

Pf: Set $N_t = M_t - M_0 \Rightarrow <N, N> = 0$.

So: N_t^2 is nonnegative c.l.m. $\Rightarrow$ it's supermart.

$E(N_t^2) \leq E(N_0^2) = 0 \Rightarrow N_t \equiv 0$ a.s. $\forall t \geq 0$.

Rmk: It means the process has no randomness.

Lemma. The limit of A_t exists (denoted by A_∞). since it's increase.

Thm. M is c.l.m. with $M_0 \in L^2$. Then:

i) M is mart bdd in L^2. $\iff E(<M, M>_\infty) < \infty$

Furthermore, if these hold, then: $M_t^2 - <M, M>_t$ is u.i

ii) M is mart. $M_t \in L^2$. $\forall t \geq 0 \iff E(<M, M>_t) < \infty$. $\forall t \geq 0$.

Furthermore, if these hold. $M_t^2 - <M, M>_t$ is mart.

Rmk: It's essential that M is mart for applying the Doob's inequility which isn't valid for c.l.m.

Pf: Replace M by $M - M_0$ to assume: $M_0 = 0$.

i) ($\Rightarrow$) By Doob's: $E(\sup_{0\leq t\leq T} M_t^2) \leq 4 E(M_T^2)$

By Fatou's: $E(\sup_{\mathbb{R}^+} M_t^2) \leq 4 \sup E(M_t^2) < \infty$.

Set $S_n = \inf\{t \geq 0 \mid \langle M,M\rangle_t \geq n\}$. Then:

$M^2_{t\wedge S_n} - \langle M,M\rangle_{t\wedge S_n}$ is dominated by $\sup M_s^2 + n$. integrable.

$\Rightarrow$ it's u.i. mart. $\therefore E(\langle M,M\rangle_{t\wedge S_n}) = E(M^2_{t\wedge S_n}) \leq E(\sup M_t^2)$

Set $n \to \infty$. then $t \to \infty$. $\Rightarrow E(\langle M,M\rangle_\infty) < \infty$.

($\Leftarrow$) Set $T_n = \inf\{t \geq 0 \mid |M_t| \geq n\}$. Then:

The c.l.m. $M^2_{t\wedge T_n} - \langle M,M\rangle_{t\wedge T_n}$ is dominated by $n^2 + \langle M,M\rangle_\infty$.

Similar argue with Fatou's $\Rightarrow (M_t)$ is bdd in L^2.

Besides. $(M_{t\wedge T_n})_{n\geq 0}$ is u.i. so converges to M_t. a.s/in L^1.

M_t is mart follows from M^{T_n} is mart.

Finally. Note: $M^2 - \langle M,M\rangle$ is dominated by $\sup M_t^2 + \langle M,M\rangle_\infty$ if these properties hold.

ii) Apply i) on $(M_{t\wedge n})_{t\geq 0}$ for every choice of n.

(4) <u>Bracket of c.l.m's:</u>

<u>Def:</u> The bracket of c.l.m's: M,N is $\langle M,N\rangle =$

$\frac{1}{2}(\langle M+N, M+N\rangle - \langle M,M\rangle - \langle N,N\rangle)$

<u>prop.</u> i) $\langle M,N\rangle$ is uniquely up to indistinguishability FV process s.t. $MN - \langle M,N\rangle$ is c.l.m.

ii) If $0 = t_0^n < t_1^n \cdots < t_{p_n}^n = t$ is subdivision of $[0,t]$ whose mesh $\to 0$. constructed by refine. Then:

$\langle M,N\rangle_t = \lim_n \sum_1^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})(N_{t_i^n} - N_{t_{i-1}^n})$ in prob.

iii) $(M,N) \longmapsto \langle M,N\rangle$ is bilinear. Symmetric.

iv) For stopping time T: $\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N^T\rangle_t$.

v) If M, N conti. mart. bdd in L^2. Then:

$M_tN_t - \langle M, N\rangle_t$ is u.i. mart. So $\langle M, N\rangle_\infty$ is well-def a.s. limit of $\langle M, N\rangle_t$ as $t\to\infty$. and satisfies: $E(M_\infty N_\infty) = E(M_0N_0) + E(\langle M, N\rangle_\infty)$.

Pf: i) Consider $(M+N)^2 - M^2 - N^2$. and uniqueness of QV.

ii)
Set $\begin{cases} X_t^n = \sum_i^{p_n} N_{t_{i-1}}(M_{t_i} - M_{t_{i-1}}) \\ \widetilde{X}_t^n = \sum_i^{p_n} M_{t_{i-1}}(N_{t_i} - N_{t_{i-1}}) \end{cases}$

$$\sum_1^j (M_{t_i} - M_{t_{i-1}})(N_{t_i^n} - N_{t_{i-1}^n}) = M_{t_j}N_{t_j} - X_{t_j}^n - \widetilde{X}_{t_j}^n$$

Similarly. show X_t^n. $\widetilde{X}_t^n$ Cauchy in L^2.

admits a subseq $\longrightarrow Y_t$. $\widetilde{Y}_t$. as and in L^1.

Define: $A_s^{(t)} = M_sN_s - Y_s - \widetilde{Y}_s$.

iii). iv) follows from ii). v) from Thm above.

Cor. $M^T(N - N^T)$ is c.l.m for c.l.m's M, N.

Pf: By iii) above.

prop. B. B' are two indept $(\mathcal{F}_t)$-BMs. Then $\langle B, B'\rangle_t = 0$. $\forall t\ge 0$.

Pf: Directly by def of $\langle B, B'\rangle$ and its bilinear.

Def: Two c.l.m's M, N is orthogonal if $\langle M, N\rangle = 0$.

Rmk: Note $\langle M, N\rangle = 0 \iff MN$ is c.l.m.

If M, N are indept c.l.m. Then so MN is c.l.m.

$\Rightarrow \langle M, N\rangle = 0$.

<u>prop.</u> (Kunita-Watanabe)

If M, N are two c.l.m's. H, K are measurable process. Then a.s:

$$\int_0^\infty |H_s K_s| \, |d\langle M,N\rangle_s| \le \left(\int_0^\infty H_s^2 \, d\langle M,M\rangle_s\right)^{\frac{1}{2}} \left(\int_0^\infty K_s^2 \, d\langle N,N\rangle_s\right)^{\frac{1}{2}}$$

<u>Pf:</u> Denote $\langle M,N\rangle_s^t = \langle M,N\rangle_t - \langle M,N\rangle_s$. $0 \le s \le t$.

1°) Prove: $|\langle M,N\rangle_s^t| \le \sqrt{\langle M,M\rangle_s^t} \sqrt{\langle N,N\rangle_s^t}$ a.s.

for $s \le t$. $s,t \in \mathbb{Q}$. (by conti. $\Rightarrow$ holds for $s,t \in \mathbb{R}^+$)

It follows from approxi. of bracket. and Cauchy.

2°) For $0 \le s \le t$. $\int_s^t |d\langle M,N\rangle_u| = \sup_{\|\pi\|} \sum |\langle M,N\rangle_{t_i}^{t_{i+1}}|$

$\le \sqrt{\langle M,M\rangle_s^t} \sqrt{\langle N,N\rangle_s^t}$. by Cauchy.

So: $\int_A |d\langle M,N\rangle_u| \le \sqrt{\int_A d\langle M,M\rangle_u} \sqrt{\int_A d\langle N,N\rangle_u}$

for $A \in \mathcal{B}_{\mathbb{R}^+}$. by MCT. argument

3°) Approxi. h, k by simple func. h_n, k_n supp on $[0,n]$.

<u>Cor.</u> For $1/p + 1/q = 1$. $E\left(\int_0^\infty |H_s K_s| \, |d\langle M,N\rangle_s|\right) \le$

$$E^{\frac{1}{p}}\left(\left(\int_0^\infty H_s^2 \, d\langle M,M\rangle_s\right)^{\frac{p}{2}}\right) E^{\frac{1}{q}}\left(\left(\int_0^\infty K_s^2 \, d\langle N,N\rangle_s\right)^{\frac{q}{2}}\right)$$

(5) <u>Conti. Semimart.</u> :

<u>Def:</u> A process $X = (X_t)$ is conti. mart. if it can be written in: $X_t = M_t + A_t$ where M is c.l.m. and A is FV process.

<u>Rmk:</u> i) The decomposition is unique up to indistinguishable.

ii) X_t has conti. sample path.

Def: Bracket of two conti. Semimart $X = M + A$ and $Y = M' + A'$ is $\langle X, Y \rangle_t$ which is defined by:

$$\langle X, Y \rangle_t = \langle M, M' \rangle_t .$$

prop. $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is subdivision of $[0,t]$ whose mesh $\to 0$. construct by refine. Then:

$$\lim_n \sum_1^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_i^n} - Y_{t_{i-1}^n}) = \langle X, Y \rangle_t \text{ in prob.}$$

Pf: $\sum_1^{p_n} \Delta X_{i-1}^i \Delta Y_{i-1}^i = \sum \Delta M_{i-1}^i \Delta M_{i-1}'^i + \sum \Delta M_{i-1}^i \Delta A_{i-1}'^i$
$+ \sum \Delta M_{i-1}'^i \Delta A_{i-1}^i + \sum \Delta A_{i-1}^i \Delta A_{i-1}'^i$

The terms involved A will vanish as $n \to \infty$

since: $\left| \sum (A_{t_i^n} - A_{t_{i-1}^n})(B_{t_i^n} - B_{t_{i-1}^n}) \right| \le$

$$\int_0^t |dA_s| \sup_i |M_{t_i^n} - M_{t_{i-1}^n}| \to 0. \ (n \to \infty)$$

$\Rightarrow$ LHS $= \langle M, M' \rangle_t = \langle X, Y \rangle_t$

Rmk: This's the reason for definition: $\langle X, Y \rangle_t$.